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DETERMINATION OF THE EXTERNAL CONTOUR OF A BODY OF REVOLUTION WITH A CENTRAL DUCT SO AS TO GIVE MINIMUM DRAG IN SUPERSONIC FLOW, WITH VARIOUS PERIMETRAL CONDITIONS IMPOSED UPON THE MISSILE GEOMETRY

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SUMMARY

Formulae and processes are presented for determining the best shape for the external surface of an annular duct in order to produce a minimum amount of wave drag in supersonic flow under certain conditions which are invoked in order to ensure that the missile geometry will obey practical design requirements. The question of what the best shape is, at least to the degree of approximation inherent in the use of the linear theory, is solved under the restriction, first, that the area between the (given) inner contour of the body of revolution and the sought outer contour is a constant, and, second, under the stipulation that the volume comprised between the surfaces which are swept out when such inner and outer contours are rotated about the duct axis is to be a constant. Two further distinctions are also made in the treatment accorded this problem on the basis of the type of inner (known) duct shape given; i.e., in one case it is assumed that the annular duct will differ but slightly from a cylinder while in the other case it is assumed that the basic shape upon which the desired external duct contour is to be built up is fundamentally a frustum of a cone. The flow through the inside of the duct is not considered at all, and the internal oblique shock is thus assumed to be attached to the entrance lip; the external contour is also assumed to "close"; i.e., the trailing edge at the duct exit does not have a blunt face. This problem had previously been solved when <u>no</u> perimetral condition was invoked, but under the more stringent conditions now being imposed it is believed that more directly useful results will be forthcoming when the numerical applications are made in another report to follow. Two distinct modes of attack are offered to attain the solution in all cases;

1. INTRODUCTION

The object of this study is to determine what is the best shape for an annular duct when it is to operate in a supersonic stream under the usual stipulation that linear theory is sufficiently accurate for the purposes in mind. In other words, the problem resolves itself into finding out what form the meridional line which delimits the external contour of the annular duct in question must have in order that the external drag due to this duct shape should be rendered a minimum, under either of the following two perimetral conditions:

(a) the area enclosed between the external meridional line defining the outside contour of the duct and the internal meridional line, which is assumed to be known beforehand, is required to have a set value.

(b) the volume, enclosed between the surfaces of revolution swept out when the above cited meridional lines are rotated about the duct's axis, is required to have a certain fixed value.

The problem will be attacked in two stages. First, the case where the radius of the circular cross-section (or parallel) representing the entrance of the duct is equal to the radius of the parallel representing the exit of the duct will be treated (see sketch "a" in Fig. 1), and then, after that, the more general case will be examined, wherein the entrance and exit cross-sections may have unequal radii (see sketch "b" in Fig. 1).

one is based on the use of Lighthill's W-function and gives the best contour shape and drag directly, while the alternative method, stemming from previously published investigations of the author, requires that a suitable distribution of supersonic sources be determined first of all, and then once the description of how these singularities vary along the x-axis is obtained, by a relatively simple process, the best contour and the related drag are derived as auxiliary information. The connection between these results and those which would be obtained by use of two-dimensional (Ackeret) theory is also pointed out as obiter dicta.

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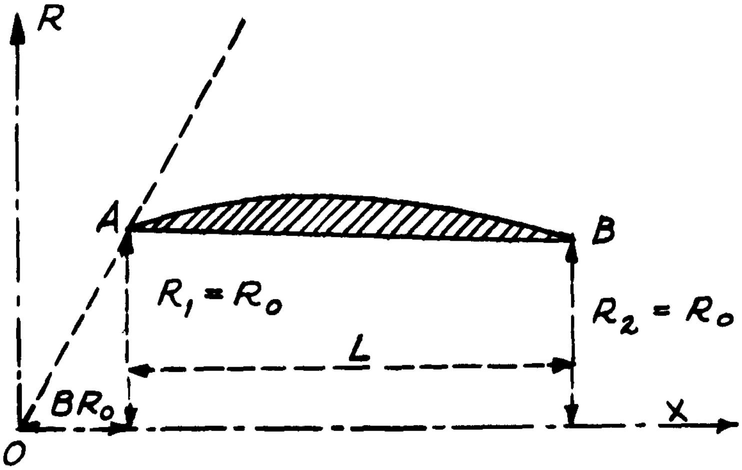
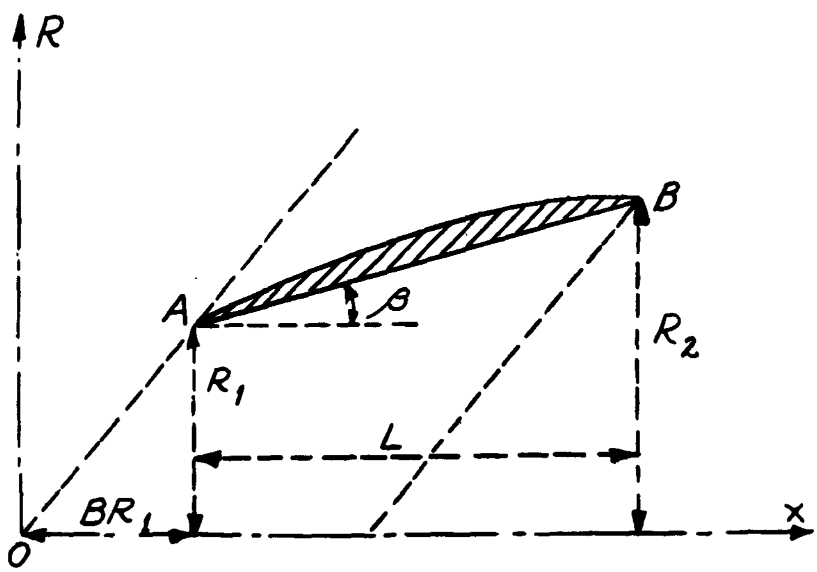


Coordinate System and Geometric Relations Applying to the Case Where the Sought Annular Duct is Cylinder-Like in Shape

Fig. 1(a)



Coordinate System and Geometric Relations Applying to the Case Where the Sought Annular Duct is Cone-Like in Shape

Fig. 1(b)

Two different approaches will be followed in obtaining the solution to the problems thus set. One will be based on the application and extension of the work done by Lighthill in Reference 1 (this way of handling the problem will constitute Part I of this study), while the other approach will be founded on the application and extension of the investigations carried out by the author previously in Reference 2 (this treatment will constitute what is called Part II of the present investigation). By use of the first of these two avenues of approach one finds what the external contour shape has to be, to give minimum drag, directly, through solution of a non-homogeneous integral equation of the second kind. In the case of the second proposed way of looking at the problem, however, one has to determine first of all what the distribution of singularities (supersonic sources) along the axis of the duct has to be, and it is found that the description of this source distribution is given by solution of an integral equation of the first kind.

2. LIST OF PRINCIPAL SYMBOLS

x = distance along the axis of the duct, measured positive in the direction of the free stream

$\vec{U}_\infty$ = free stream velocity vector

R = Perpendicular distance away from the duct axis

R_1 = radius of the inlet section of the duct

R_2 = radius of the outlet section of the duct

R_o = $\dfrac{R_1 + R_2}{2}$

R_i = radius of the inner circular section, produced when the annular duct is cut by the plane x = constant

R_e = radius of the outer circular section, produced when the annular duct is cut by the same plane x = constant

β = $\dfrac{R_2 - R_1}{L}$

M_∞ = Mach number of the free stream flow with velocity U_∞

B = $\sqrt{M_\infty^2 - 1}$

s = $\dfrac{x}{B\,R_1} - 1$, $s_2 = \dfrac{x}{B\,R_2}$

r = $\dfrac{R_e}{B\,R_o}$, $r_1 = \dfrac{R_e}{B\,R_1}$, $r_2 = \dfrac{R_e}{B\,R_2}$

L = length of the duct, measured along its axis

ℓ = $\dfrac{L}{BR_o}$, $\ell_1 = \dfrac{L}{BR_1}$, $\ell_2 = \dfrac{L}{BR_2}$

η = $\dfrac{dr_1}{ds} = \dfrac{dR_e}{dx}$

$f(x)$ = local strengths of the supersonic source distribution which is placed along the axis of the duct, and which is responsible for the creation of the perturbed flow which flows past and in conformity with the external contour of the annular duct

p = local pressure

p_∞ = pressure corresponding to a point located in the uniform undisturbed free-stream flow

ρ = local density

ρ_∞ = density corresponding to a point located in the uniform undisturbed free-stream flow

γ = adiabatic exponent; equals 1.405 for air

C_D = drag coefficient = $\dfrac{\text{drag exerted on the exterior surface of the duct}}{\frac{1}{2}\rho_\infty U_\infty^2 R_0^2}$

PART I

3. Case of an Annular Duct with Cylindrical Type of Central Channel ($R_1 = R_2 = R_0$). Solution by Extension of Lighthill's Method

The drag coefficient that results from determination of the sum of all the resultant forces created by action of the pressure increments, $p - p_\infty$, that are exerted against the external surface of this sort of duct has been obtained by Lighthill in Reference 1 and the pertinent formula has the form

$$C_D = 4\int_0^{\ell} \eta^2 \, dz - 2\int_0^{\ell}\int_0^{\ell} W(|s - z|)\, \eta(s)\, ds \; dz \tag{1}$$

where the function $W(y)$, as a variable that is dependent on the assigned y-values, has been evaluated by the Admiralty Computing Service, and the tabulated results have been given in the reference by Lighthill just mentioned; they are reproduced here in the appended Table I.

TABLE I

Lighthill's W(y) Function

y	W(y)	y	W(y)
0.0	0.50000	3.6	0.06929
0.2	0.43190	3.8	0.06384
0.4	0.37552	4.0	0.05896
0.6	0.32889	4.4	0.05058
0.8	0.28867	4.8	0.04370
1.0	0.25497	5.2	0.03803
1.2	0.22617	5.6	0.03331
1.4	0.20143	6.0	0.02936
1.6	0.18009	6.4	0.02602
1.8	0.16155	6.8	0.02320
2.0	0.14542	7.2	0.02078
2.2	0.13131	7.6	0.01869
2.4	0.11889	8.0	0.01688
2.6	0.10798	8.4	0.01534
2.8	0.09833	8.8	0.01397
3.0	0.08976	9.2	0.01277
3.2	0.08216	9.6	0.01170
3.4	0.07536	10.0	0.01077

Let attention now be turned to the perimetral condition that may be stated as follows: in any arbitrary general one of the meridional half-planes passing through the axis of the body of revolution under discussion, the area that is to be enclosed between the meridional line delimiting the outer boundary of the annular body and the meridional line marking out the inner boundary of the duct is to turn out to have a given fixed value. This condition is expressible by the simple relation

$$\int_{BR_o}^{BR_o + L} (R_e - R_i)\, dx = constant$$

On the other hand if it is taken for granted that $R_i = R_i(x)$ is known, then this constraining condition will have the form

$$\int_0^{\ell} (r - \tfrac{1}{B})\, dz = \int_0^{\ell} dz \int_0^{z} \eta(s)\, ds = C \tag{2}$$

while it is required, in addition, because of the assumption that $R_1 = R_2 = R_o$, that

$$\int_0^{\ell} \eta(z)\, dz = 0 \tag{3}$$

must hold true.

Looking back at Eq. (1) now, it is seen that when an arbitrary variation $\Delta\eta$ is given to the variable η the first variation in C_D (which will be denoted here by the symbol ΔC_D) will take on the form

$$\Delta C_D = \int_0^{\ell} \left[8\eta - 4F(z) \right] \Delta\eta(z)\, dz \tag{4}$$

provided one uses the shorthand notation that

$$F(z) = \int_0^{\ell} W(|s - z|)\, \eta(s)\, ds \tag{5}$$

Moreover, since conditions (2) and (3) must always be satisfied, the variation in η, denoted by $\Delta\eta$, must obey the restrictions

and

$$\left.\begin{aligned} &\int_0^\ell \Delta\eta(z)\,dz = 0 \\ &\int_0^\ell dz \int_0^z \Delta\eta(s)\,ds = 0 \end{aligned}\right\} \tag{6}$$

If the order of the integrations indicated in the last of the restraining conditions given as Eqs. (6) is now inverted, and if due advantage is taken of the relationship set down as the first of the Eqs. (6), then the perimetral condition becomes just

$$\int_0^\ell z\,\Delta\eta(z)\,dz = 0 \tag{7}$$

If the slope function $\eta = \eta(z)$ really constitutes the description of the meridional line contour which produces a minimum drag coefficient, C_D, then the first variation ΔC_D has to turn out to be zero whatever the variation of η, $\Delta\eta$, is, so long as it satisfies the conditions stated mathematically by means of Eqs. (6) and (7). Consequently, if λ_1 and λ_2 are taken to be two quite arbitrary constants, a priori, it follows that the relation given below must be satisfied:

$$8\eta - 4F(z) + \lambda_1 + \lambda_2 z = 0 \tag{8}$$

If, on the other hand, the perimetral condition imposed on the shape of the external duct contour being sought is the one described in the (b) part of the first paragraph in the Introduction, this situation is translated into analytic language by writing that

$$\int_{BR_o}^{BR_o+L} (R_e^2 - R_i^2)\,dx = 2\int_{BR_o}^{BR_o+L} \frac{R_e+R_i}{2}(R_e-R_i)\,dx = \text{constant}$$

must be invoked.

Now, since in the case under consideration one has approximately that

$$\frac{R_e + R_i}{2R_o} \cong 1 \tag{9}$$

it is seen that once again the relation

$$\int_0^{\ell} \left(r - \frac{1}{B}\right) d\zeta = constant \tag{2'}$$

holds, which is identical with condition (2).

It is evident, therefore, that to the degree of approximation premised by writing the simplification given as Eq. (9), the perimetral conditions stated in the Introduction as stipulations (a) and (b) are both represented mathematically by the same equation, to all intents and purposes of the study now to be undertaken. Of course, it is true that there does exist a difference between the actual constants which appear on the right hand sides of Eqs. (2) and (2'), but there is, none-the-less, no difference in the general form of these two equations of restraint.

It is worth while to reiterate that Eq. (8) delineates precisely the meridional line corresponding to a body which has least drag. As a matter of fact, the second variation, $\Delta_2 C_D$, of the drag coefficient, C_D, resulting from an arbitrary variation in η, of $\Delta\eta$, may be written in the form

$$\Delta_2 C_D = 4\int_0^{\ell} \left[\Delta\eta(\zeta)\right]^2 d\zeta - 2\int_0^{\ell}\int_0^{\ell} W(|s-\zeta|)\,\Delta\eta(s)\,\Delta\eta(\zeta)\,ds\,d\zeta > 0 \tag{10}$$

by reference once again to Eq. (1).

Now if the conditions stated as Eqs. (6) and (7) are still to be satisfied, then $\Delta_2 C_D$ corresponds to the drag coefficient that is produced by an annular duct having a value for R_o that is the same as the inlet section and the same as the outlet section, as well, and whose meridional line is described by means of

the function $\Delta \eta$ (z). The drag coefficient of this duct, however, is always going to be positive.

4. Solution of the Variational Equation, Eq. (8)

The solution of Eq. (8) is carried out by replacing the indicated function F(z) which appears in this Eq. (8) by its longer expression in terms of Lighthill's W-function obtained from Eq. (5). When this is done, one has that

$$\eta(z) = \frac{1}{2} \int_0^{\ell} W(|s - z|)\, \eta(s)\, ds + \lambda_1 + \lambda_2 z \qquad (11)$$

where the two new constants standing on the right hand side of this equation are still being represented by means of the symbols λ_1 and λ_2 that had formerly been employed in Eq. (8).

The expression now obtained as Eq. (11) constitutes a Fredholm integral equation. Its solution may be written in the standard form

$$\eta(z) = \lambda_1 + \lambda_2 z + \frac{1}{2} \int_0^{\ell} \Gamma(z, s)\, (\lambda_1 + \lambda_2 s)\, ds \qquad (12)$$

where the resolvent kernel Γ (z,s) is defined in terms of a Fredholm series, which is uniformly convergent regardless of the value of ℓ. It will normally be true, however, that, in those cases of practical interest in the solution of actual problems such as are under examination here, the values of ℓ will be such that one may replace the Fredholm series by a Neumann series and reap the no mean benefit of sharply reducing the tediousness of the involved calculations which would otherwise result.

Thus it will be to advantage to express the resolvent kernel Γ (z,s) in terms of the series

$$\Gamma(z, s) = K(z, s) + \frac{1}{2} K^{(2)}(z, s) + \cdots + \frac{1}{2^{n-1}} K^{(n)}(z, s) + \cdots \qquad (13)$$

where
$$\left.\begin{aligned} K(z,s) &= W(|s-z|) \\ K^{(2)}(z,s) &= \int_0^{\ell} K(z,t)\,K(t,s)\,dt \\ &\cdots \\ K^{(n)}(z,s) &= \int_0^{\ell} K(z,t)\,K^{(n-1)}(t,s)\,dt \end{aligned}\right\} \qquad (14)$$

In order for the Neumann series, Eq. (13), to be uniformly convergent, it is sufficient to stipulate, on the basis of the condition given by E. Schmidt in Reference 3, that the relation

$$\frac{1}{2} < \frac{1}{\sqrt{\int_0^{\ell}\int_0^{\ell}\left[W(|s-z|)\right]^2 ds\,dz}} \qquad (15)$$

should hold.

Now in the present case it is true that

$$W(|s-z|) < 0.5 + \frac{A-0.5}{\ell}\ |s-z| \qquad (16)$$

where $W(\ell)$ has been denoted by A. If one remembers that for $\ell > 4$ the corresponding value of A will be so small that it can be neglected, then this value, appearing on the right hand side of Eq. 16, which is larger than $W(|s-z|)$, may be substituted in place of $W(|s-z|)$ in Eq. (15), so that, in fact the value of the double integral appearing in Eq. (15) will be bounded by the quantity $\ell^2/8$; that is,

$$\int_0^{\ell}\int_0^{\ell}\left[W(|s-z|)\right]^2 ds\,dz << \frac{\ell^2}{8}$$

and thus it follows that the condition (15) may be recast into the simple statement that the Neumann series, Eq. (13), will be guaranteed to converge provided that

$$\ell < 2\sqrt{8} = 5.656. \qquad (17)$$

Of course, the radius of convergence of the Neumann series, Eq. (13), is somewhat larger in actuality than is allowed on the basis of the limiting value given in Eq. (17). For large values of ℓ , however, it will be more convenient to look for a solution of Eq. (11) in finite terms; this can be done by replacing the kernel $W(|s-z|)$ by a trigonometric series having a finite number of terms. If the series is to have m+1 terms, then at each of m+1 selected points along the $|s-z|$- axis lying within the interval from 0 to ℓ this series will be required to take on the same value as the W function has there. Thus, in symbolic form, the kernel will be expressed as

$$W(|s-z|) = \sum_{n=0}^{m} A_n \cos\left[\frac{n\pi}{\ell}(z-s)\right] \tag{18}$$

wherein the coefficients A_n are determined according to the formula of Clairaut - De la Vallée Poussin[4] as

$$\left.\begin{aligned} A_n &= \frac{2}{m}\sum_{r=1}^{m-1} W\left(\frac{r\ell}{m}\right)\cos\frac{r\pi n}{m} + \frac{W(0)+(-1)^n W(\ell)}{m} \\ &\qquad \text{for } n = 0, 1, \cdots, m-1 \\ \text{and} \quad A_m &= \frac{1}{2}\cdot\frac{2}{m}\sum_{r=1}^{m-1}(-1)^r W\left(\frac{r\ell}{m}\right) + \frac{W(0)+(-1)^m W(\ell)}{m} \end{aligned}\right\} \tag{19}$$

If the expression for $W(|s-z|)$ given by means of Eq. (18) is now substituted into Eq. (11), and if the auxiliary definitions are made that

$$\theta = \frac{\pi z}{\ell} \;;\; \theta_1 = s\frac{\pi}{\ell} \;;\; \text{and } \ell^* = \frac{\ell}{2\pi} \tag{20}$$

then Eq. (11) is transformed into

$$\eta(\theta) = \lambda_1 + 2\lambda_2\ell^*\theta + \ell^*\int_0^{\pi}\sum_{n=0}^{m} A_n(\cos n\theta\cos n\theta_1 + \sin n\theta\sin n\theta_1)\,\eta(\theta_1)\,d\theta_1 \tag{11'}$$

$$= \lambda_1 + 2\lambda_2\ell^*\theta + \ell^*\left[\sum_{n=0}^{m} A_n H_n\cos n\theta + \sum_{n=1}^{m} A_n K_n\sin n\theta\right]$$

wherein H_n and K_n are constants which are defined by the relations

$$\begin{aligned} H_n &= \int_0^{\pi} \eta\,(\theta_1)\cos n\theta_1\, d\theta_1 \\ K_n &= \int_0^{\pi} \eta\,(\theta_1)\sin n\theta_1\, d\theta_1 \end{aligned} \tag{21}$$

In order to evaluate these constants, one may multiply both sides of the expression for $\eta(\theta)$, given as Eq. (11'), by $\cos n\,\theta_1 d\theta_1$ and $\sin n\theta_1 d\theta_1$, respectively, and integrate between the limits of 0 and π. For convenience's sake, when carrying out this operation, one may make the definition that

$$\left.\begin{aligned} B_n &= \int_0^{\pi} (\lambda_1 + 2\lambda_2\, \ell^*\theta)\cos n\theta\, d\theta \\ \text{and}\quad C_n &= \int_0^{\pi} (\lambda_1 + 2\lambda_2\, \ell^*\theta)\sin n\theta\, d\theta \end{aligned}\right\} \tag{22}$$

and thus $B_o = \pi\,(\lambda_1 + \lambda_2\, \ell^*\pi)$

$$\text{and}\ \begin{cases} B_n = -\frac{4}{n^2}\lambda_2\ell^* & \text{for n odd} \\ \quad = 0 & \text{for n even} \end{cases}$$

$$\text{while}\ \begin{cases} C_n = \frac{2}{n}\lambda_1 + 2\lambda_2\ell^*\frac{\pi}{n} & \text{for n odd} \\ \quad = -\frac{\pi}{n}2\lambda_2\ell^* & \text{for n even.} \end{cases}$$

Once these understandings have been agreed upon, it will be seen that the H and K constants are related as follows:

$$\left.\begin{aligned} b_r H_r &= B_r + \ell^* \sum_i A_i K_i\left(\frac{1}{i+r} - \frac{1}{r-i}\right) \\ b_i K_i &= C_i + \ell^* \sum_r A_r H_r\left(\frac{1}{i+r} - \frac{1}{r-i}\right) \quad \text{for } r = 0,2,4,\cdots \\ &\qquad\qquad \text{and } i = 1,3,5,\cdots \end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned} b_i H_i &= B_i + \ell^* \sum_r A_r K_r\left(\frac{1}{i+r} - \frac{1}{i-r}\right) \\ b_r K_r &= C_r + \ell^* \sum_i A_i H_i\left(\frac{1}{i+r} - \frac{1}{i-r}\right) \\ &\qquad \text{for } i = 1,3,5,\cdots \\ &\qquad \text{and } r = 2,4,6,\cdots \end{aligned}\right\} \tag{23'}$$

wherein

$$\left.\begin{aligned} b_o &= 1 - \pi \ell^* A_o \\ b_r &= 1 - \frac{\pi}{2} \ell^* A_r \qquad \text{for } r \neq 0 \\ \text{and} \quad b_i &= 1 - \frac{\pi}{2} \ell^* A_i . \end{aligned}\right\} \tag{24}$$

It is evident, therefore, that the set of 2m+1 equations in m+1 unknowns H and K may be separated into two independent systems of equations, one of which serves to determine the unknowns H_r and K_i , while the other allows the determination of the unknowns H_i and K_r.

This decomposition into groups of similarly determined quantities may be carried even a step farther, inasmuch as the second of the Eqs. (23) may be written as

$$K_i = \frac{C_i}{b_i} + \ell^* \sum_r A_r \frac{b_{ri}}{b_i} H_r \tag{25}$$

where another simplification has been made by setting

$$b_{ri} = \frac{1}{i+r} - \frac{1}{r-i} \tag{26}$$

and thus if Eq. (25) is substituted into the first of the two relationships given as Eq. (23) it follows that

$$b_r H_r = B_r + \ell^* \sum_i A_i C_i \frac{b_{ri}}{b_i} + \ell^{*2} \sum_i A_i b_{ri} \sum_t A_t H_t \frac{b_{ti}}{b_i}$$

$$\text{for } t = 0,2,4,\cdots \qquad (23'')$$

$$r = 0,2,4,\cdots$$

$$\text{and } i = 1,3,5,\cdots .$$

This latter relationship now permits the determination of all the H-values for which the subscripts are even, and after these quantities are computed, then the second equation in the set given as Eq. (23) may be utilized to find the values of all the K-values with odd subscripts.

In an entirely analogous manner, if one first makes the simplifying definition that

$$b_{ir} = \frac{1}{i+r} - \frac{1}{i-r} \tag{26'}$$

it will be found, by working with Eq. (23'), that all the H-values having odd-numbered subscripts are given by means of the relation

$$b_i H_i = B_i + \ell^* \sum_r A_r C_r \frac{b_{ir}}{b_r} + \ell^{*2} \sum_r A_r b_{ir} \sum_t A_t \frac{b_{it}}{b_r} H_t$$

for $t = 1,3,5,\cdots$

$r = 0,2,4,\cdots$

and $i = 1,3,5,\cdots$.

Once these quantities have been determined, then all the K-values having even-numbered subscripts may be found by having recourse to the second of the set of relationships given as Eq. (23').

Since λ_1 and λ_2 are to be quite arbitrary constants, then it is worthy of note that the method just described will no longer be applicable in that case where ℓ^* might take on that value for which $\pi \ell^* A_0 = 1$. If this were to happen, it would have to be true that this critical value of ℓ must be such as to satisfy the condition

$$\int_0^\ell W(z)\, dz = 2 \tag{27}$$

where it is taken for granted that W(z) is to be expressed in this instance by Eq. (18) and where it then follows that A_0 is to be evaluated from the formula

$$A_0 = \frac{1}{\ell} \int_0^\ell W(z)\, dz .$$

Now it merely remains to note that there is no value of ℓ for which the condition stated as Eq. (27) will be satisfied; this fact may be verified by checking through the values of Table I, which is, in effect, a definition of the function W(z). This non-existence of a suitable behavior for the W(z) function

in this special case is all the more reason, therefore, why the other b_i and b_r coefficients will not drop out whatever the value of ℓ may be, and consequently one may conclude that the method being proposed here is, in truth, always going to be applicable.

Finally, it is to be remarked that Eqs. (23) and (23') are written in such a way as to permit the determination of the H_r, K_i set of constants on the one hand and the set H_i, K_r on the other hand by means of an iteration process, provided that the numerical values for the coefficients multiplying the unknown constants are such as to ensure that the method itself will converge.

5. Determination of the Constants λ_1 and λ_2

The results derived in the preceding Article will now permit one to write the solution for $\eta(z)$ in the form

$$\eta(z) = \lambda_1 F_1(z) + \lambda_2 F_2(z) \tag{28}$$

where F_1 and F_2 are known functions.

Thus, one is led to the conclusion under these circumstances that

$$r - \frac{1}{B} = \int_0^z \eta(z)\,dz = \lambda_1 \int_0^z F_1(z)\,dz + \lambda_2 \int_0^z F_2(z)\,dz \tag{29}$$

and, therefore, by applying the condition of "closure" given as Eq. (3), it follows that

$$\lambda_1 \int_0^\ell F_1(z)\,dz + \lambda_2 \int_0^\ell F_2(z)\,dz = 0 \tag{3*}$$

and meanwhile it is also true that the perimetral condition originally stated as Eq. (2) will result in the further restriction that

$$\lambda_1 \int_0^\ell dz \int_0^s F_1(s)\,ds + \lambda_2 \int_0^\ell dz \int_0^s F_2(s)\,ds = C \tag{2*}$$

These conditions (2^*) and (3^*) constitute the definitive requirements which will serve to determine the constants λ_1 and λ_2 uniquely. After their evaluation has been made in this way, then there will be no further trouble in applying the result given as Eq. (29) to derive the sought description of the meridional line directly.

6. Case of an Annular Duct with Conical Type of Central Channel ($R_1 \neq R_2$). Extension of Lighthill's Method to Determination of the Flow About a Conical Type of Duct, Before Optimization

First of all, before tackling the problem of determining the best duct shape in this case, it is essential to give prior consideration to the problem of how one needs to go about finding the flow created over the outside of such a conical sort of annular duct which arises from action of a uniform supersonic stream that is assumed to be impinging upon it, under the simple assumption that $R_1 \neq R_2$; i.e., where the radius of the circular cross-section at the duct entrance is R_1 and the radius of the circular cross-section at the duct exit is R_2, and where these two cross-sections are substantially different in size. In order to carry out such a calculation a procedure will be utilized which is quite analogous to the one employed by Lighthill[1] in his study of the similar problem which is concerned with the determination of the flow about a nearly cylindrical sort of annular duct.

To this end, one may start the analysis of such a flow problem by studying the disturbed field that is produced by a distribution of supersonic sources located along the x-axis with local strengths per unit distance of $U_\infty f(\xi)$. It is well-known that the body's geometric properties are linked to the source strengths under these circumstances by application of the boundary conditions

which are the same for any and all arbitrary general meridional planes through the body axis. Thus it may be stated that

$$\frac{dR_e}{dx} \cong \frac{1}{R_e} \int_0^{x-BR_e} \frac{(x-\xi)\,\dot{f}(\xi)\,d\xi}{\sqrt{(x-\xi)^2 - B^2 R_e^2}} \tag{30}$$

where $\dot{f}(\xi) = \frac{df}{d\xi}$.

Now if β is used to denote the semi-vertex angle of the cone which forms the basic shape upon which the actual exterior contour is to be built up, it will be quite legitimate in the present situation to let

$$R_e = R_1 + \beta(x - BR_1).$$

If, furthermore, the following changes in notation are made for purposes of casting the above expression into more tractable form:

$$\left.\begin{aligned} \xi &= BR_1 t \\ \frac{1}{B}\frac{dr_1}{dz} &= g(z) \\ \text{and}\quad \dot{f}(\xi) &= h(t) \end{aligned}\right\} \tag{31}$$

then the integral relationship written as Eq. (30) becomes converted into the expression

$$g(z) = \frac{1}{1+\beta Bz} \int_0^{(1-\beta B)z} \frac{(1+z-t)\,h(t)\,dt}{\sqrt{(1+z-t)^2 - (1+\beta Bz)^2}} \tag{32}$$

or it is even simpler when written as

$$g^*(z) = (1+\beta Bz)g(z) = \int_0^{(1-\beta B)z} \frac{(1+z-t)\,h(t)\,dt}{[(1-\beta B)z - t][(1+\beta B)z - t + 2]}$$

If one now multiplies both sides of this latter equation by $\frac{dz}{\sqrt{s-z}}$ and integrates between the limits of 0 and s , it follows that

$$v(s) = \int_0^s \frac{g^*(z)}{\sqrt{s-z}}\,dz = \int_0^{(1-\beta B)s} h(t)\,dt \int_{\frac{t}{1-\beta B}}^{s} \frac{(z+1-t)\,dz}{\sqrt{[(1-\beta B)z-t][(1+\beta B)z-t+2]}} \cdot \frac{1}{\sqrt{s-z}} \tag{32'}$$

as a result of interchanging the order of integration with respect to t and z.

Upon carrying out the process of integration with respect to z which is indicated on the right hand side of this Eq. (32') it is found that

$$v(s) = \frac{\pi}{\sqrt{2}} \int_0^{(1-\beta B)s} h(t) \sqrt{\frac{1+\frac{\beta B}{1-\beta B}\, t}{1-\beta^2 B^2}}\; K\left(\frac{s - \frac{t}{1-\beta B}}{1 + \frac{\beta B t}{1-\beta B}} \right) dt \tag{33}$$

where

$$\frac{\pi}{2\sqrt{2}} K(x) = \sqrt{\frac{2}{1+\beta B} + x}\;\; \mathbb{E}\left(\sqrt{\frac{x}{x+\frac{2}{1+\beta B}}} \right) - \frac{1-\beta B}{1+\beta B}\; \frac{\mathbb{F}\left(\sqrt{\frac{x}{x+\frac{2}{1+\beta B}}} \right)}{\sqrt{x+\frac{2}{1+\beta B}}} \tag{34}$$

in which the complete elliptic integrals of the first and second kinds have been denoted by the symbols $\mathbb{F}$ and $\mathbb{E}$, respectively.

On the basis of this definition for $K(x)$ given as Eq. (34) it is seen that $K(0) = \sqrt{1+\beta B}$. In addition, let the following auxiliary definitions be made for convenience's sake:

$$\begin{aligned} t_1 &= \frac{t}{1-\beta B} \\ h\left[s\,(1-\beta B)\right] &= h^*(s) \\ h\left[t_1\,(1-\beta B)\right] &= h^*(t_1) \end{aligned} \tag{35}$$

and $$w(s) = \frac{1}{\sqrt{1-\beta B}} \quad \frac{\sqrt{2}}{\pi} \quad \frac{v'(s)}{\sqrt{1-\beta B s}}$$

It then follows simply, by taking the derivative with respect to s of both sides of Eq. (33), that

$$w(s) = h^*(s) + \frac{1}{\sqrt{1-\beta B}} \int_0^s h^*(t_1)\, \frac{\dot{K}\left(\frac{s-t_1}{1+\beta B t_1}\right)}{\sqrt{1-\beta B s}\,\sqrt{1+\beta B t_1}}\, dt_1 \tag{36}$$

where the dot has the usual significance, i.e., $\dot{K}(x) = \frac{dK}{dx}$.

Now it so happens that Eq. (36) is a Volterra type of integral equation of

the second kind, and its solution may be written down at once as

$$h^*(s) = w(s) - \frac{1}{\sqrt{1+\beta B}} \int_0^s \Gamma(s, t_1)\, w(t_1)\, dt_1 \tag{37}$$

wherein the resolvent kernel is given by the following series, which is once again uniformly convergent (5) in this case:

$$\Gamma(s, t_1) = H^*(s,t_1) - \frac{1}{\sqrt{1+\beta B}} H^{*(2)}(s,t_1) + \frac{1}{1+\beta B} H^{*(3)}(s,t_1) - \cdots \tag{38}$$

wherein the several H^* symbols have been substituted for the following more complex expressions:

$$H^*(s, t_1) = \frac{\dot{K}\left(\frac{s-t_1}{1+\beta B t_1}\right)}{\sqrt{1+\beta B s}\ \sqrt{1+\beta B t_1}} = \frac{H\left(\frac{s-t_1}{1+\beta B t_1}\right)}{\sqrt{1+\beta B s}\ \sqrt{1+\beta B t_1}}$$

$$H^{*(2)}(s, t_1) = \int_{t_1}^{s} H^*(s, u)\, H^*(u, t_1)\, du$$

$$H^{*(3)}(s, t_1) = \int_{t_1}^{s} H^{*(2)}(u, t_1)\, H^*(s, u)\, du$$

and so on.

Now, further consider the first of these integral relationships; in more extended form, one has that

$$H^{*(2)}(s, t_1) = \int_{t_1}^{s} \frac{H\left(\frac{s-u}{1+\beta B u}\right)}{\sqrt{1+\beta B s}\ \sqrt{1+\beta B u}} \cdot \frac{H\left(\frac{u-t_1}{1+\beta B t_1}\right)}{\sqrt{1+\beta B u}\ \sqrt{1+\beta B t_1}}\, du$$

and if a change of variable is made in which one lets

$$\frac{u-t_1}{1+\beta B t_1} = u_1$$

then the above expression for $H^{*(2)}(s,t_1)$ becomes

$$H^{*(2)}(s,t_1) = \int_0^{(s-t_1)/1+\beta Bt_1} \frac{H(u_1)\, H\left(\frac{s-t_1}{1+\beta Bt_1}\cdot\frac{1}{1+\beta Bu_1}-\frac{u_1}{1+\beta Bu_1}\right)du_1}{\sqrt{1+\beta Bs}\,\sqrt{1+\beta Bt_1}\,(1+\beta Bu_1)}$$

$$= \frac{1}{\sqrt{1+\beta Bs}}\cdot\frac{1}{\sqrt{1+\beta Bt_1}}\int_0^{y}\frac{H(u_1)H\left(\frac{y}{1+\beta Bu_1}-\frac{u_1}{1+\beta Bu_1}\right)du_1}{1+\beta Bu_1}$$

$$= \frac{1}{\sqrt{1+\beta Bs}}\cdot\frac{1}{\sqrt{1+\beta Bt_1}}\; H_2\left(\frac{s-t_1}{1+\beta Bt_1}\right)$$

where the symbol y has been employed momentarily to stand for the relation $y = \frac{s-t_1}{1+\beta Bt_1}$.

In an entirely analogous way, it will be seen that

$$H^{*(3)}(s,t_1) = \int_{t_1}^{s} \frac{H\left(\frac{s-u}{1+\beta Bu}\right)}{\sqrt{1+\beta Bs}\,\sqrt{1+\beta Bu}}\cdot\frac{H_2\left(\frac{u-t_1}{1+\beta Bt_1}\right)}{\sqrt{1+\beta Bu}\,\sqrt{1+\beta Bt_1}}\,du$$

$$= \frac{1}{\sqrt{1+\beta Bs}}\cdot\frac{1}{\sqrt{1+\beta Bt_1}}\int_0^{y}\frac{H_2(u_1)H\left(\frac{y}{1+\beta Bu_1}-\frac{u_1}{1+\beta Bu_1}\right)du_1}{1+\beta Bu_1}$$

$$= \frac{1}{\sqrt{1+\beta Bs}}\cdot\frac{1}{\sqrt{1+\beta Bt_1}}\; H_3\left(\frac{s-t_1}{1+\beta Bt_1}\right)$$

and likewise for the others.

On the basis of these evaluations, it follows, therefore, that the value of the resolvent kernel, $\Gamma(s,t_1)$, may be obtained from the series

$$\Gamma(s,t_1) = \frac{1}{\sqrt{1+\beta Bs}\,\sqrt{1+\beta Bt_1}}\left[H\left(\frac{s-t_1}{1+\beta Bt_1}\right)-\frac{1}{\sqrt{1+\beta B}}H_2\left(\frac{s-t_1}{1+\beta Bt_1}\right)\right.$$
$$\left.+\frac{1}{1+\beta B}\; H_3\left(\frac{s-t_1}{1+\beta Bt_1}\right)-\cdots\right] \tag{38'}$$
$$= \frac{1}{\sqrt{1+\beta Bs}\,\sqrt{1+\beta Bt_1}}\;\Gamma^*\left(\frac{s-t_1}{1+\beta Bt_1}\right)$$

and upon insertion of this result into Eq. (37) it is found finally that

$$h^*(s) = w(s) - \frac{1}{\sqrt{1+\beta B}} \int_0^{s} \frac{\Gamma^*\left(\frac{s-t_1}{1+\beta B t_1}\right)}{\sqrt{1+\beta B s}\sqrt{1+\beta B t_1}} w(t_1)\, dt_1 \qquad (37')$$

7. <u>Computation of the Pressures and Resultant Forces Acting on the Cone-Like Body</u>

The component of velocity, lying in the direction of the x-axis, which is due to the disturbed flow about the body in question, is given at any arbitrary general point on the surface of the cone (which is characterized by the fact that the given circular entrance and exit openings of the annular duct under study constitute directrices of this conical surface) by means of the usual formula:

$$-\frac{1}{U_\infty}\frac{\partial\phi}{\partial x} = \int_0^{x-BR_e} \frac{\dot{f}(\xi)\, d\xi}{\sqrt{(x-\xi)^2 - B^2 R_e^2}}$$

$$= \int_0^{(1-B\beta)\eta} \frac{h(t)\, dt}{\sqrt{(1+\eta-t)^2 - (1+\beta B\eta)^2}}$$

$$= \int_0^{\eta} \frac{h^*(t_1)\, dt_1}{\sqrt{(\eta - t_1)\left[\frac{1+\beta B}{1-\beta B}\eta - t_1 + \frac{2}{1-\beta B}\right]}}$$

where the free-stream velocity, U_∞, is used for non-dimensionalizing purposes.

By insertion of the value for $h^*(t_1)$, obtainable through means of Eq. (37), the expression for this x-component of the perturbed velocity field over the conical surface (as compared with the free-stream velocity) is obtained as

$$-\frac{1}{U_\infty}\frac{d\Phi}{dx} = \int_0^z \frac{w(t_1)\,dt_1}{\sqrt{(z-t_1)\left(\frac{1+\beta B}{1-\beta B}z - t_1 + \frac{2}{1-\beta B}\right)}}$$

$$-\frac{1}{\sqrt{1-\beta B}}\int_0^z \frac{dt_1}{(z-t_1)\left(\frac{1+\beta B}{1-\beta B}z - t_1 + \frac{2}{1-\beta B}\right)}\int_0^{t_1}\Gamma(t_1,u)w(u)\,du$$

$$= \int_0^z \frac{w(u)\,du}{\sqrt{(z-u)\left(\frac{1+\beta B}{1-\beta B}z - u + \frac{2}{1-\beta B}\right)}}$$

$$-\frac{1}{\sqrt{1+\beta B}}\int_0^z w(u)\,du\int_u^z \frac{\Gamma(t_1,u)\,dt_1}{\sqrt{(z-t_1)\left(\frac{1+\beta B}{1-\beta B}z - t_1 + \frac{2}{1-\beta B}\right)}}$$

$$= \int_0^z w(u)\,T(z,u)\,du$$

where the symbol $T(z, u)$ employed in this integral stands for the mixed expression

$$T(z,u) = \frac{1}{\sqrt{(z-u)\left(\frac{1+\beta B}{1-\beta B}z - u + \frac{2}{1-\beta B}\right)}} \tag{39}$$

$$-\frac{1}{\sqrt{1+\beta B}}\int_u^z \frac{\Gamma(t_1,u)\,dt_1}{\sqrt{(z-t_1)\left(\frac{1+\beta B}{1-\beta B}z - t_1 + \frac{2}{1-\beta B}\right)}}\ .$$

Now with a slight amount of further manipulation it is seen that

$$w(u) = \frac{\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{1-\beta B}}\quad\frac{\dot{v}(u)}{\sqrt{1+\beta Bu}}$$

$$= \frac{\sqrt{2}}{\pi}\ \frac{1}{\sqrt{1-\beta B}}\cdot\frac{1}{\sqrt{1+\beta Bu}}\left[\frac{g^*(0)}{\sqrt{u}} + \int_0^u \frac{\dot{g}^*(z)\,dz}{\sqrt{u-z}}\right]$$

where, as usual, the dot over a function serves to indicate that the derivative of the function so marked has been taken with respect to the sole parameter upon which it depends.

It thus results that the sought x-component of velocity is given by the relation

$$-\frac{1}{U_\infty}\frac{\partial \phi}{\partial x} = \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{1-\beta B}}\left[g^*(o)\int_o^{z} \frac{T(z,u)}{\sqrt{u}\ \sqrt{1+\beta B u}}\,du + \int_o^{z}\frac{T(z,u)\,du}{\sqrt{1+\beta B u}}\int_o^{u}\frac{\dot g^*(s)}{\sqrt{u-s}}\,ds\right] \tag{40}$$

Now the second term on the right hand side of this Eq. (40) may be further reduced by interchanging the order of integration, so that in its rearranged form it becomes

$$\frac{\sqrt{2}}{\pi}\ \frac{1}{\sqrt{1-\beta B}}\int_o^{z}\frac{T(z,u)du}{\sqrt{1+\beta B u}}\int_o^{u}\frac{\dot g^*(s)}{\sqrt{u-s}}\,ds$$

$$= \frac{\sqrt{2}}{\pi}\ \frac{1}{\sqrt{1-\beta B}}\int_o^{z}\dot g^*(s)\,ds\int_s^{z}\frac{T(z,u)\,du}{\sqrt{u-s}\ \sqrt{1+\beta B u}}$$

$$= \int_o^{z}\dot g^*(s)\ Z(z,s)\,ds$$

where Z (z,s) is the integral:

$$Z(z,s) = \int_s^{z}\frac{T(z,u)}{\sqrt{u-s}\ \sqrt{1+\beta B u}}\ \frac{\sqrt{2}}{\pi}\ \frac{1}{\sqrt{1-\beta B}}\ . \tag{41}$$

On the basis of this definition it is obvious that one has, in addition, that

$$Z(z,o) = \frac{\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{1-\beta B}}\int_o^{z}\frac{T(z,u)}{\sqrt{u}\ \sqrt{1+\beta B u}}\,du$$

and thus one may write the expression for the sought x-component of velocity as

$$-\frac{1}{U_\infty}\frac{\partial \phi}{\partial x} = g^*(o)\,Z(z,o) + \int_o^{z}\dot g^*(s)\ Z(z,s)\,ds$$
$$= \int_{-\infty}^{z} Z(z,s)\ d\left[g^*(s)\right]. \tag{40'}$$

Further information about the function $Z\ (z,s)$ may be obtained by inserting into Eq. (41) the expression for the $T(z,u)$ function given as Eq. (39). When this is done, it turns out that

$$Z(z,s) = \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{1-\beta B}}\Bigg[\int_s^z \frac{du}{\sqrt{u-s}\ \sqrt{1-\beta Bu}\ \sqrt{(z-u)\left(\frac{1+\beta B}{1-\beta B}z-u+\frac{2}{1-\beta B}\right)}}$$

$$-\frac{1}{\sqrt{1+\beta B}}\int_s^z \frac{du}{\sqrt{u-s}\ \sqrt{1+\beta Bu}}\int_u^z \frac{\Gamma(t_1,u)\,dt_1}{\sqrt{z-t_1}\ \sqrt{\frac{1+\beta B}{1-\beta B}z-t_1+\frac{2}{1-\beta B}}}\Bigg]$$

$$= \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{1-\beta B}}\Bigg[\int_s^z \frac{du}{\sqrt{u-s}\ \sqrt{1-\beta Bu}\ \sqrt{(z-s)\left(\frac{1+\beta B}{1-\beta B}z-u+\frac{2}{1-\beta B}\right)}}$$

$$-\frac{1}{\sqrt{1+\beta B}}\int_{\frac{z-s}{1+\beta Bs}}^{0}\frac{\Gamma^*(\tau)\,d\tau}{\sqrt{1+\beta B\tau}}\int_s^{\frac{z-\tau}{1+\beta B\tau}}\frac{du}{\sqrt{u-s}\sqrt{1+\beta Bu}\sqrt{\left[z-u-\tau(1+\beta Bu)\right]\left[\frac{1+\beta B}{1-\beta B}z-u-\tau(1+\beta Bu)+\frac{2}{1-\beta B}\right]}}\Bigg]$$

provided that one momentarily makes use of the new variable, defined by the relation $\tau = \dfrac{t_1-u}{1+\beta Bu}$.

Now also let the following new definitions be set up for simplicity's sake:

$$k^2 = \frac{\left(\frac{1}{\beta B}+\frac{1+\beta B}{1-\beta B}z+\frac{2}{1-\beta B}\right)(z-s)}{\left(z+\frac{1}{\beta B}\right)\left(\frac{1+\beta B}{1-\beta B}z+\frac{2}{1-\beta B}-s\right)}$$

$$m = \frac{2}{\sqrt{\beta B}}\ \frac{1}{\sqrt{\left(z+\frac{1}{\beta B}\right)\left(\frac{1+\beta B}{1-\beta B}z+\frac{2}{1-\beta B}-s\right)}}$$

$$k_1^2 = \frac{\left(\frac{z}{1-\beta B}-\frac{\tau}{1+\beta B}+\frac{2}{1-\beta^2B^2}+\frac{1}{\beta B}\right)\left(\frac{z-\tau}{1+\beta B}-s\right)}{\left(\frac{z-\tau}{1+\beta B}+\frac{1}{\beta B}\right)\left(\frac{z}{1-\beta B}-\frac{\tau}{1+\beta B}+\frac{2}{1-\beta^2B^2}-s\right)} \tag{42}$$

and

$$m_1 = \frac{2}{\sqrt{\beta B}}\ \frac{1}{\left(\frac{z}{1-\beta B}-\frac{\tau}{1+\beta B}+\frac{2}{1-\beta^2B^2}-s\right)\left(\frac{z-\tau}{1+\beta B}+\frac{1}{\beta B}\right)}\ .$$

It then follows that the expression for $Z\ (z,s)$ reduces to

$$Z(z,s) = \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{1-\beta B}}\left[m\,F(k)+\frac{1}{\sqrt{1+\beta B}}\int_0^{\frac{z-s}{1+\beta Bs}}\frac{\Gamma^*(\tau)}{\sqrt{1+\beta B\tau}}\,m_1F(k_1)\,d\tau\right]. \tag{43}$$

Then if one notes that

$$Z(z,z) = \frac{\sqrt{2}}{\pi}\,\frac{1}{\sqrt{1-\beta B}}\cdot\frac{\sqrt{2}\,\sqrt{1-\beta B}}{1+\beta B_z}\cdot\frac{\pi}{2} = \frac{1}{1+\beta B_z} \qquad (44)$$

it is easily perceived that the x-component of the velocity in the disturbed field may be rewritten, from the formulation obtained as Eq. (40'), into the quite compact statement

$$\begin{aligned} -\frac{1}{U_\infty}\frac{\partial \phi}{\partial x} &= g^*(z)\frac{1}{1+\beta B_z} - \int_0^z S(z,s)\,g^*(s)\,ds \\ &= g(z) - \int_0^z S(z,s)\,g^*(s)\,ds \end{aligned} \qquad (45)$$

where the partial derivative of Z with respect to s is denoted by

$$\frac{\partial Z}{\partial s} = S(z,s). \qquad (46)$$

Corresponding to this result for determination of the x-component of the disturbed velocity field, the following expression for the local pressure coefficient holds:

$$\begin{aligned} \frac{p-p_\infty}{\rho_\infty V_\infty^2} &= g(z) - \int_0^z S(z,s)\,g^*(s)\,ds \\ &= \frac{1}{B}\left[\eta(z) - \int_0^z S_1(z,s)\,\eta(s)\,ds\right] \end{aligned} \qquad (47)$$

where an improvement in brevity has been wrought by setting $S_1(z,s) = (1+\beta B_s)S(z,s)$.

Consequently, the drag coefficient produced by this sort of flow acting on the conical duct is given by

$$\begin{aligned} C_D &= 4\,\frac{R_1^2}{R_0^2}\left[\int_0^{l_1}(1+\beta B_z)\eta^2(z)\,dz - \int_0^{l_1}(1+\beta B_z)\eta(z)\,dz\int_0^{l_1} S(z,s)(1+\beta B_s)\eta(s)\,ds\right] \\ &= \frac{R_1^2}{R_0^2}\left[4\int_0^{l_1}(1+\beta B_z)\eta^2(z)\,dz - 2\int_0^{l_1}(1+\beta B_z)\eta(z)\,dz\int_0^{l_1} S^*\eta(s)(1+\beta B_s)\,ds\right] \end{aligned} \qquad (48)$$

$$
\left.\begin{array}{lll}
\text{where} \quad S^* = S(z,s) & \text{for} & 0 \le s \le z \\
\text{while} \quad S^* = S(s,z) & \text{for} & z \le s \le \ell_1 \;.
\end{array}\right\} \qquad (49)
$$

8. <u>Setting Up of the Equation Which Will Determine the External Shape of the Conical-Like Body Giving the Minimum Wave Drag</u>

By use of the expression (48) just derived, it will be possible to set up a variational equation which will allow the determination of the required meridional line $\eta(z)$. This is done in an exactly analogous way to what was done in the case of the cylinder-like duct treated in Article 3. The sought meridional line is the one which will produce a cone-like annular duct having minimum drag, and which will be subject to the same conditions as stated in Article 1 in the form of the perimetral conditions (a) or (b). The stipulation stated as restraining condition (a) is converted into mathematical form in the present instance in just exactly the same way that was met with in establishing Eq. (2), so that here also one has that

$$
\int_0^{\ell_1} dz \int_0^{z} \eta(s)\, ds = C \qquad (2'')
$$

must be satisfied, while the formerly stated perimetral condition given as Eq. (2'), having to do with the volume enclosed, takes on an altered form in this present case, because now it is required that

$$
\int_0^{\ell_1} (1+\beta B z)\, dz \int_0^{z} \eta(s)\, ds = D = \text{constant} \qquad (50)
$$

where the approximation

$$
\frac{R_e + R_i}{2} \cong R_i + \beta\,(x - BR_1) = R_1\,(1+\beta B z)
$$

has been utilized once again just as it was in writing Eq. (32).

It must also be true that, whatever the variation in the contour of the duct happens to be, the further restriction stated as

$$\int_{BR_1}^{BR_1+L} \frac{dR_e}{dx}\, dx = R_2 - R_1 = R_1 \beta B \ell_1$$

must be invoked; this "closure" condition on the contour may be simplified to read simply that

$$\int_0^{\ell_1} \eta(z)\, dz = E = \text{constant} \tag{50'}$$

must hold.

When an arbitrary variation to η, again denoted by $\Delta\eta$, is brought about, a first variation in the drag coefficient, C_D, again denoted by ΔC_D, will result, and it will have the form

$$\Delta C_D = \frac{R_1^2}{R_0^2}\Bigg\{ 8\int_0^{\ell_1}(1+\beta B z)\,\eta(z)\,\Delta\eta(z)\,dz$$
$$-2\Bigg[\int_0^{\ell_1}(1+\beta B z)\,\eta(z)\,dz\int_0^{\ell_1} S^*\Delta\eta(s)(1+\beta B s)\,ds + \int_0^{\ell_1}(1+\beta B z)\Delta\eta(z)\,dz\int_0^{\ell_1}(1+\beta B s)S^*\eta(s)\,ds\Bigg]\Bigg\}.$$

The terms appearing within the square brackets in this expression may be combined in the following manner:

Firstly, the order of integration may be interchanged in the first of these two integrals, to give

$$\int_0^{\ell_1}(1+\beta B z)\,\eta(z)\,dz\int_0^{\ell_1}(1+\beta B s)\,S^*\Delta\eta(s)\,ds =$$
$$\int_0^{\ell_1}(1+\beta B s)\,\Delta\eta(s)\,ds\int_0^{\ell_1}(1+\beta B z)\,S^*\eta(z)\,dz$$

where one must remember that, in the evaluation of the integral $\int_0^{\ell_1} (1+\beta B z) S^* \eta(z)\, dz$, it will be necessary to make use of the convention previously stated as Eq. (49); i.e., it is agreed that

$$S^* = S(s,z) \qquad \text{for } 0 \leq z \leq s$$
$$\text{while } \quad = S(z,s) \qquad \text{for } s \leq z \leq \ell_1 .$$

Upon interchange of the s and z symbols, it then follows that

$$\int_0^{\ell_1} (1+\beta B s)\, \Delta\eta(s)\, ds \int_0^{\ell_1} (1+\beta B z)\, S^*\, \eta(z)\, dz = \int_0^{\ell_1} (1+\beta B z)\, \Delta\eta(z)\, dz \int_0^{\ell_1} (1+\beta B s)\, S^*\, \eta(s)\, ds$$

in which the convention must now be observed that

$$S^* = S\,(z,s) \qquad \text{for } 0 \leq s \leq z$$
$$\text{while } \quad = S\,(s,z) \qquad \text{for } z \leq s \leq \ell \ .$$

Consequently, it then turns out that

$$\Delta C_D = \frac{R_1^2}{R_0^2} \left\{ 8 \int_0^{\ell_1} (1+\beta B z)\eta(z)\Delta\eta(z) dz - 4 \int_0^{\ell_1} (1+\beta B z)\Delta\eta(z) \int_0^{\ell_1} (1+\beta B s) S^* \eta(s)\, ds \right\}$$
$$= \frac{R_1^2}{R_0^2} \left\{ \int_0^{\ell_1} (1+\beta B z) \Big[8\eta(z) - 4F(z) \Big] \Delta\eta(z)\, dz \right\} \tag{51}$$

provided it is understood that

$$F(z) = \int_0^{\ell_1} (1+\beta B s)\, S^*\, \eta(s)\, ds\, . \tag{52}$$

On the other hand, on the basis of the restraints given as Eqs. (2") and (50')

it follows that the variation must also satisfy the conditions:

$$\int_0^{l_1} dz \int_0^{z} \Delta\eta(s)\, ds = 0$$

and $$\int_0^{l_1} \Delta\eta(z)\, dz = 0$$

or else the simplified versions

$$\int_0^{l_1} z\,\Delta\eta(z)\, dz = 0 \qquad \text{and} \qquad \int_0^{l_1} \Delta\eta(z)\, dz = 0 \tag{53}$$

must be invoked, while, if the perimetral condition on the volume enclosed between the duct's inner and outer surface, as stated by Eq. (50), is to be the one governing the variational procedure, it will be required that

$$\int_0^{l_1} (1+\beta B z)\, dz \int_0^{z} \Delta\eta(s)\, ds = 0$$

or, by interchanging the order of integration once again, this minimum volume perimetral condition states that

$$\int_0^{l_1} \left(z + \frac{\beta B z^2}{2}\right) \Delta\eta(z)\, dz = 0 \tag{53'}$$

must hold.

If the first variation in C_D, denoted by ΔC_D, is now to be made to vanish by proper selection of $\eta(z)$, so that regardless of the variation in η, denoted by $\Delta\eta(z)$ [so long as Eq. (53) is satisfied] it must be true that

$$8(1+\beta B z)\,\eta(z) - 4(1+\beta B z)\,F(z) + \lambda_1 + \lambda_2 z = 0 \tag{54}$$

is to hold, while, on the other hand, if the constraining conditions to be satisfied by the variation, $\Delta\eta(z)$, are the ones stated as Eq. (53') together with the second one of the set given as Eq. (53), it must be true that

$$8(1+\beta B z)\,\eta(z) - 4(1+\beta B z)\,F(z) + \lambda_1 + \lambda_2' z\left(1+\frac{\beta B}{2} z\right) = 0 \tag{54'}$$

holds.

9. Solution of the Integral Equation (54) [or Eq. (54')] Giving the Contour That Will Produce a Minimum Drag

If the expression for $F(z)$, stated as Eq. (52), is now inserted into Eq. (54) one obtains the relationship for $\eta(z)$ in the form

$$\eta(z) = \frac{1}{2}\int_0^{l_1} (1+\beta B s)\, S^{*}\eta(s)\, ds + \frac{\lambda_1 + \lambda_2 z}{1+\beta B z} \tag{54"}$$

which is a Fredholm integral equation, and it is the basic determining relationship which, together with the conditions written above as Eq. (53) [or Eq. (53')], serves to define the sought function for the meridional line producing least drag, $\eta(z)$, and which also determines the constants λ_1 and λ_2 (or λ_2').

The solution of this Fredholm integral equation may once more be represented, just as was done in Article 3, by means of the following expression

$$\eta(z) = \frac{\lambda_1 + \lambda_2 z}{1+\beta B z} + \frac{1}{2}\int_0^{l_1} \Gamma(z,s)\, \frac{\lambda_1 + \lambda_2 s}{1+\beta B s}\, ds \tag{55}$$

wherein the resolvent kernel, denoted by $\Gamma(z,s)$, is to be handled in just the same way that the analogous situation was attacked when the solution to Eq. 11, given as Eq. (12), was encountered in identical form in Article 4.

A solution to Eq. (54") may actually also be obtained directly by proceeding according to the method propounded by Goursat[6]. This procedure will afford a treatment of the present problem which will be recognized as being quite similar to the one employed previously in Article 4.

First of all, let

$(1+\beta B z)\,\eta(z) = \varphi(z)$ and $(1+\beta B s)\,\eta(s) = \varphi(s)$

and then rewrite Eq. (54") in the form

$$\varphi(z) = \frac{1}{2}(1+\beta B z)\int_0^{l_1} S^{*}\varphi(s)\, ds + \lambda_1 + \lambda_2 z. \tag{56}$$

Now, in addition, set $\theta = \frac{\pi z}{\ell_1}$ and $\theta_1 = \frac{\pi s}{\ell_1}$ and let S^{**} be represented (approximately) by means of the trigonometric series of a finite number (2m + 1) of terms in the form

$$S^{**} = \sum_{n=0}^{m} A_n \cos n\theta \cos n\theta_1 + \sum_{k=1}^{m} A_{m+k} \sin k\theta \sin k\theta_1 \qquad (57)$$

where the constants A_n and A_{m+k} are determined by applying the condition that the sum of the squares of the deviations between S^* and S^{**}, throughout the interval from 0 to π, is to be a minimum; i.e., the quantity E^* is to be a minimum, where

$$E^* = \int_0^{\pi} \int_0^{\pi} \left[K_1(\theta, \theta_1) \right]^2 d\theta \, d\theta_1 \qquad (57')$$

and where the difference between S^* and S^{**} is defined by

$$K_1(\theta, \theta_1) = S^* - S^{**}.$$

If the expression for S^{**} given as Eq. (57) is inserted into the above expression for $K_1(\theta, \theta_1)$, the resulting expression for E^* will be seen to be a function of the A's only, and the condition that E^* should be a minimum (or maximum) is given by the statement that

$$\frac{\partial E^*}{\partial A} = 0 \qquad (57'')$$

must hold true.

Now let the following convenient notation be introduced:

$$\epsilon_n = \int_0^\pi \int_0^\pi S^* \cos n\theta \cos n\theta_1 \, d\theta \, d\theta_1 \qquad \text{for } n = 0,1,2,\cdots,m$$

$$\epsilon_{m+k} = \int_0^\pi \int_0^\pi S^* \sin k\theta \sin k\theta_1 \, d\theta \, d\theta_1 \qquad \text{for } k = 1,2,3,\cdots,m$$

$$\mu_{n,i} = \int_0^\pi \int_0^\pi \cos n\theta \cos n\theta_1 \cos i\theta \cos i\theta_1 \, d\theta \, d\theta_1$$

$$\left.\begin{aligned} &= \frac{\pi^2}{4} \text{ for } n = i \\ &= 0 \text{ for } n \neq i \end{aligned}\right\} \text{if } i \leq m$$

$$\mu_{n,m+k} = \int_0^\pi \int_0^\pi \cos n\theta \cos n\theta_1 \sin k\theta \sin k\theta_1 \, d\theta \, d\theta_1 \qquad (58)$$

$$= \frac{4k^2}{(k^2-u^2)^2} \qquad \text{for } k + n \text{ odd}$$

(and thus also for $k - n$ odd)

$$= 0 \qquad \text{for } k + n \text{ even}$$

(and thus also for $k - n$ even)

$$\mu_{m+k,n} = \mu_{n,m+k}$$

and $\mu_{m+k,\,m+j} = \frac{\pi^2}{4}$ if $k = j$ while $\mu_{m+k,\,m+j} = 0$ if $k \neq j$

Writing out the conditions for an extremal, Eq. (57"), in explicit form, one obtains the following relationships, where m is taken to be <u>even</u>, for instance:

$$\left.\begin{aligned}
\epsilon_0 &= \mu_{0,0}A_0 + \mu_{0,m+1}A_{m+1} + \mu_{0,m+3}A_{m+3} + \cdots + \mu_{0,2m-1}A_{2m-1} \\
\epsilon_1 &= \mu_{1,1}A_1 + \mu_{1,m+2}A_{m+2} + \mu_{1,m+4}A_{m+4} + \cdots + \mu_{1,2m}A_{2m} \\
\epsilon_3 &= \mu_{2,2}A_2 + \mu_{2,m+1}A_{m+1} + \cdots \qquad + \mu_{2,2m-1}A_{2m-1} \\
&\cdot \\
&\cdot \\
&\cdot \\
\epsilon_{m+1} &= \mu_{m+1,0}A_0 + \mu_{m+1,2}A_2 + \cdots + \mu_{m+1,m}A_m + \mu_{m+1,m+1}A_{m+1} \\
\epsilon_{m+2} &= \mu_{m+2,1}A_1 + \mu_{m+2,3}A_3 + \cdots \mu_{m+2,m-1}A_{m-1} + \mu_{m+2,m+2}A_{m+2}
\end{aligned}\right\} \quad (59)$$

The set of equations just written as Eq. (59) may be separated into two independent systems of equations. One of these systems contains m+1 individual members of the type given by

$$\left.\begin{aligned}
&\epsilon_n = \mu_{n,n}A_n + \sum_{k=1}^{m-1} k\,\mu_{n,m+k}A_{m+k} \\
&\text{and} \\
&\epsilon_{m+k} = \mu_{m+k,m+k}A_{m+k} + \sum_{n=0}^{m} \mu_{m+k,n}A_n \qquad \text{for } n = 0,2,\cdots,m \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } k = 1,3,\cdots,m-1
\end{aligned}\right\} \quad (59')$$

The other independent set of equations contains m individual members of the type given by

$$\left.\begin{aligned}
&\epsilon_i = \mu_{i,i}A_i + \sum_{j=2}^{m} \mu_{i,m+j}A_{m+j} \\
&\text{and} \\
&\epsilon_{m+j} = \mu_{m+j,m+j}A_{m+j} + \sum_{i=1}^{m-1} \mu_{m+j,i}A_i \qquad \text{for } i = 1,3,\cdots,m-1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } j = 2,4,\cdots,m
\end{aligned}\right\} \quad (59'')$$

and, what is more, these above two sets of equations can be reduced to the fol-

lowing two independent sets, one of which will have $\frac{m}{2}+1$ individual members, and is given by

$$\epsilon_n = \mu_{n,n} A_n + \sum_{k=1}^{m-1} \mu_{n,m+k} \left[\frac{\epsilon_{m+k}}{\mu_{m+k,m+k}} - \sum_{\nu=0}^{m} \frac{\mu_{m+k,\nu}}{\mu_{m+k,m+k}} A_\nu \right] \tag{60}$$

for $n = 0,2,\cdots,m$
$\nu = 0,2,\cdots,m$
and $k = 1,3,\cdots,m-1$

while the other will have only $\frac{m}{2}$ individual members, and is given by

$$\epsilon_i = \mu_{i,i} A_i + \sum_{j=2}^{m} \mu_{i,m+j} \left[\frac{\epsilon_{m+j}}{\mu_{m+j,m+j}} - \sum_{p=1}^{m-1} \frac{\mu_{m+j,p}}{\mu_{m+j,m+j}} A_p \right] \tag{60'}$$

for $i = 1,3,\cdots,m-1$
$j = 2,4,\cdots,m$
and $p = 1,3,\cdots,m-1$.

It is worth pointing out at this point that a very significant property that is characteristic of the E's may be seen to be brought out by the array of equations given above as Eqs. (58); that is, it is quite easy to deduce from them that the computation of the E's may be performed without even the necessity of having to know what S^* is explicitly.

As a matter of fact, in order to give some elaboration on this statement, let attention be focussed first of all on the distribution of supersonic sources which are to be located along the x-axis and which will then produce the required radial component of velocity given by the formula

$$U_\infty \eta_n(z) = U_\infty \frac{\cos n\theta}{1+\beta B \frac{l_1}{\pi}\theta}$$

at each point, located in the interval $BR_1 \leq x \leq L+BR_1$, and lying on the generatrix of the cone which is determined by the entrance and exit openings of the

annular duct under consideration. Also, let ϕ_n represent the potential which describes this flow. Then the relationship given as Eq. (37') may be used to find out how the strength of the source distribution must vary along the x-axis in order to satisfy the imposed boundary condition at the surface of the duct, or, on the other hand, one may employ the approximate procedure described by von Kármán and Moore in Reference 7 to effect the same end more simply.

The corresponding drag coefficient, that is defined in the conventional manner as

$$(C_D)_n = \frac{-2\pi \rho_\infty U_\infty \int_{BR_1}^{L+BR_1} \frac{1}{U_\infty} \frac{\partial \phi_n}{\partial x} R \eta_n(z)\, dx}{\pi R_0^2 \cdot \frac{1}{2} \rho_\infty U_\infty^2}$$

$$= -4B \int_0^{\ell_1} \frac{1}{U_\infty} \frac{\partial \phi_n}{\partial x} (1+\beta B z)\, \eta_n(z)\, dz \tag{61}$$

may consequently be determined without further trouble.

On the other hand, on the basis of a comparison of Eq. (58) with Eq. (48), it would appear that this drag coefficient must also be given by

$$(C_D)_n = \frac{R_1^2}{R_0^2}\left[\frac{4\ell_1}{\pi} \int_0^{\pi} \frac{\cos^2 n\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}\, d\theta - 2\, \frac{\ell_1^2}{\pi^2}\, \epsilon_n \right]. \tag{61'}$$

Of course, one may evaluate the first term within the square brackets by means of the following reduction formula:

$$\int_0^{\pi} \frac{\cos^2 n\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}\, d\theta = \frac{1}{2a} \cos\frac{2n}{a}\left[C_i\left(\frac{2n}{a} + 2n\pi\right) - C_i\left(\frac{2n}{a}\right) - \log(1+a\pi) \right]$$
$$+ \frac{1}{2a} \sin\frac{2n}{a}\left[S_i\left(\frac{2n}{a} + 2n\pi\right) - S_i\left(\frac{2n}{a}\right) \right]$$

where $a = \beta B \frac{\ell_1}{\pi}$, and C_i and Δ_i are the sine-integral function and cosine-integral function, respectively. Thus, it follows that Eq. (61') may be used to obtain the values of ϵ_n directly.

In an entirely analogous manner, one may obtain the ϵ_{m+k} values. Beginning with the statement of the condition on the radial velocity components produced by the distribution of supersonic sources placed along the x-axis; i.e., using the relationship that

$$U_\infty \; \eta_{m+k} = U_\infty \frac{\sin k\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}$$

one may obtain the value of $(C_D)_{m+k}$ directly by means of a formula that is analogous to Eq. (61) where $(C_D)_{m+k}$ is the corresponding drag coefficient that is defined in the usual conventional manner in just the same way as was done previously. This drag coefficient is thus obtained as

$$(C_D)_{m+k} = \frac{R_1^2}{R_0^2}\left[\frac{4\ell_1}{\pi}\int_0^\pi \frac{\sin^2 k\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}\, d\theta - \frac{2\ell_1^2}{\pi^2}\epsilon_{m+k}\right] \qquad (61'')$$

and, in this case, the evaluation of the first term may be performed by means of the extended formula

$$\int_0^\pi \frac{\sin^2 k\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}\, d\theta = \frac{1}{2a}\cos\frac{2k}{a}\left[-C_i\left(\frac{2k}{a}+2k\pi\right)+C_i\left(\frac{2k}{a}\right)+\log(1+a\pi)\right]$$
$$-\frac{1}{2a}\sin\frac{2k}{a}\left[\Delta_i\left(\frac{2k}{a}+2k\pi\right)-\Delta_i\left(\frac{2k}{a}\right)\right]$$

so that it again follows that Eq. (61") may be employed to give the sought values of ϵ_{m+k} ‡.

---

‡ It may be of interest to point out that the first term on the right hand side of Eq. (61') [or on the right hand side of Eq. (61")] will give the value

(continued on next page)

Returning now to the main trend of this argument it will be assumed that the two independent systems of equations, Eq. (60) and Eq. (60'), have been solved, and that the values of A_n and A_{m+k} have been calculated, once the approximate value S^{**} has been accepted as a suitable representation of the quantity S^*. After this has been done, then the determining equation for eliciting the required meridional contour may be converted from the original statement given above as Eq. (56) over into the new form which appears now as:

$$\varphi(\theta) = \lambda_1 + \lambda_2 \frac{\ell_1}{\pi}\theta + \frac{\ell_1}{2\pi}\left(1 + \beta B \frac{\ell_1}{2\pi}\theta\right)\int_0^{\pi} S^{**}\varphi(\theta_1)\,d\theta_1$$

$$= \lambda_1 + 2\lambda_2 \ell_1^*\theta + \ell_1^*(1 + 2\beta B \ell_1^*\theta)\left[\sum_{n=0}^{m} A_n H_n \cos n\theta + \sum_{k=1}^{m} A_{m+k} K_k \sin k\theta\right] \qquad (62)$$

where the new symbols appearing herein are defined as

$$\left.\begin{aligned} \ell_1^* &= \frac{\ell_1}{2\pi} \\ H_n &= \int_0^{\pi} \varphi(\theta_1)\cos n\theta_1\,d\theta_1 \\ K_k &= \int_0^{\pi} \varphi(\theta_1)\sin k\theta_1\,d\theta_1 \end{aligned}\right\} \qquad (63)$$

with obvious analogy to the definitions made earlier in Eq. (21).

---

† of $(\overline{C}_D)$, which is the value that C_D would have if, in the calculation of this drag coefficient, one were to employ the formula which gives the pressure under the condition that flow is assumed to be two-dimensional. In other words, it turns out that the ϵ-values are proportional to the difference between the value of the drag coefficient $(\overline{C}_D)$, obtained in the manner just mentioned, and the drag coefficient which would result from a calculation based on magnitudes for the pressures which would be obtained by working with the correct formula, given as Eq. (47).

Now let the following useful constants B_n and C_n be defined as follows:

$$B_n = \int_0^{\pi} (\lambda_1 + 2\ell_1^* \lambda_2 \theta) \cos n\theta \, d\theta = -2\ell_1^* \lambda_2 \frac{2}{n^2}, \text{ for } n \text{ odd}$$

$$= \lambda_1 \pi + \lambda_2 \ell_1^* \pi^2, \text{ for } n = 0$$

$$= 0 \qquad , \text{ for } n \text{ even}$$

and

$$C_n = \int_0^{\pi} (\lambda_1 + 2\ell_1^* \lambda_2 \theta) \sin n\theta \, d\theta = \frac{2}{n} \lambda_1 + 2\ell_1^* \lambda_2 \frac{\pi}{n}, \text{ for } n \text{ odd}$$

$$= -2\ell_1^* \lambda_2 \frac{\pi}{n}, \text{ for } n \text{ even}$$

$$= 0 \qquad \text{ for } n = 0$$

with obvious analogy to the way the similar quantities were defined in Eq. (22).

Consequently, by utilizing these definitions it is seen that the following relationships between the H and K values are obtained on the basis of what was previously stated as Eq. (62):

and

$$\left.\begin{aligned} c_r H_r &= B_r + \ell_1^* \sum_i b_{ri} A_{m+i} K_i + 2\beta B \ell_1^{*2} L_r \\ c_i' K_i &= C_i + \ell_1^* \sum_r b_{ri} A_{m+r} H_r + 2\beta B \ell_1^{*2} L_i' \\ c_i H_i &= B_i + \ell_1^* \sum_r b_{ir} A_{m+r} K_r + 2\beta B \ell_1^{*2} L_i \\ c_r' K_r &= C_r + \ell_1^* \sum_i b_{ir} A_{m+i} H_i + 2\beta B \ell_1^{*2} L_r' \end{aligned}\right\} \quad (65)$$

for $r = 0, 2, 4, \cdots$

and $i = 1, 3, 5, \cdots$

where

$$
\left.\begin{aligned}
c_0 &= 1 - \ell_1^* \pi A_0 - \beta B \ell_1^{*2} \pi^2 A_0 \\
c_r &= 1 - \ell_1^* \frac{\pi}{2} A_r - \beta B \ell_1^{*2} \pi A_r \\
c_i &= 1 - \ell_1^* \frac{\pi}{2} A_i - \beta B \ell_1^{*2} \pi A_i \\
c_r' &= 1 - \ell_1^* \frac{\pi}{2} A_{m+r} - \beta B \ell_1^{*2} \pi A_{m+r} \\
\text{and} \quad c_i' &= 1 - \ell_1^* \frac{\pi}{2} A_{m+i} - \beta B \ell_1^{*2} \pi A_{m+i}
\end{aligned}\right\} \quad (66)
$$

while

$$
\left.\begin{aligned}
b_{ri} &= \frac{1}{i+r} - \frac{1}{r-i} \\
b_{ir} &= \frac{1}{i+r} - \frac{1}{i-r} \\
L_r &= -\sum_i{}' A_i H_i \left[\frac{1}{(i+r)^2} + \frac{1}{(i-r)^2} \right] \\
&\quad + \sum_j{}' A_{m+j} K_j \left[-(-1)^{r+j} \frac{\pi}{2} b_{rj} \right] - A_{m+r} K_r \frac{\pi}{2r} \\
L_i' &= -\sum_r{}' A_{m+r} K_r \left[\frac{1}{(i+r)^2} + \frac{1}{(i-r)^2} \right] \\
&\quad + \sum_j{}' A_j H_j \left[-(-1)^{i+j} \frac{\pi}{2} b_{ji} \right] - A_i H_i \frac{\pi}{2i} \\
L_i &= -\sum_r{}' A_r H_r \left[\frac{1}{(i+r)^2} + \frac{1}{(i-r)^2} \right] \\
&\quad + \sum_j{}' A_{m+j} K_j \left[-(-1)^{i+j} \frac{\pi}{2} b_{ij} \right] - A_{m+i} K_i \frac{\pi}{2i} \\
\text{and} \quad L_r' &= -\sum_i{}' A_{m+i} K_i \left[\frac{1}{(i+r)^2} + \frac{1}{(i-r)^2} \right] \\
&\quad + \sum_j{}' A_j H_j \left[-(-1)^{j+r} \frac{\pi}{2} b_{jr} \right] - A_r H_r \frac{\pi}{2r} \\
&\qquad \text{for} \quad j = 0, 1, 2, \cdots, m
\end{aligned}\right\} \quad (66')
$$

in which the prime on the summation symbol is for the purpose of pointing out that the term which has both of its subscripts the same is to be dropped when the terms are summed.

It is admittedly true that the calculation of the H and K values by means of Eq. (65) will be somewhat more complex than in the corresponding case encountered when dealing with Eqs. (23) and (23'), but nevertheless it is to be remarked that Eq. (65) is admirably suited to treatment by an iteration process for the purpose of finding the H and K values, and this iterative procedure must surely converge if β is small enough. The successive steps in such a process will consist of, first, finding a preliminary evaluation for H and K [let them be called $H^{(1)}$ and $K^{(1)}$] by dropping out the L-terms in Eq. (65), and then proceeding to find a second approximation by utilization of values for the L's which are obtained by inserting into Eq. (66') the approximate values $H^{(1)}$ and $K^{(1)}$ in place of the actual H and K values. Once the first approximate values of L are obtained, then Eq. (65) can be solved once again to give the next more accurate evaluation for the H and K values, and this process may be kept up until the required accuracy is achieved.

10. Determination of the Constants λ_1 and λ_2

In order to obtain the values of the constants λ_1 and λ_2 one may do just what was done previously in Article 5. That is to say, the solution of Eq. (54) [or of Eq. (54')] is written in the form

$$\eta(z) = \lambda_1 F_1(z) + \lambda_2 F_2(z) \tag{67}$$

wherein F_1 and F_2 are definitely known functions.

Then here again one may deduce from the above statement that

$$r_1 - \frac{1}{B} = \int_0^{z} \eta(z)\,dz - \lambda_1 \int_0^{z} F_1(z)\,dz + \lambda_2 \int_0^{z} F_2(z)\,dz$$

and besides, for geometrical consistency, it is required that

$$\frac{1}{B}\,\frac{R_2 - R_1}{R_1} = \lambda_1 \int_0^{\ell_1} F_1(z)\,dz + \lambda_2 \int_0^{\ell_1} F_2(z)\,dz \tag{68}$$

while, because of the perimetral condition stated as Eq. (2"), it must be true that

$$\lambda_1 \int_0^{\ell_1} dz \int_0^{z} F_1(s)\,ds + \lambda_2 \int_0^{\ell_1} dz \int_0^{z} F_2(s)\,ds = C. \tag{69}$$

Similarly, it follows, because of the perimetral condition stated previously as Eq. (50) that one must impose the condition that

$$\lambda_1 \int_0^{\ell_1} (1 + \beta B z)\,dz \int_0^{z} F_1(s)\,ds + \lambda_2' \int_0^{\ell_1} (1 + \beta B z)\,dz \int_0^{z} F_2(s)\,ds = D \tag{70}$$

if, this time, the volume enclosed between the duct's inner and outer surfaces is to be made a minimum. Thus, this Eq. (70), together with Eq. (68), wherein the constant λ_2' has been substituted in place of λ_2, will afford the means of determining the constants under consideration in this alternative situation.

PART II

1. Direction to be Taken in This New Approach

The problem which was stated and investigated in Part I can also be solved by making use of a mode of attack which is quite similar to the one used by the author when he previously had occasion to study the problem of determining the solid missile shape which would give a minimum drag, the solution for which is given in Reference 2. This new alternative way of treating the problem now under consideration differs from the approach used in Part I essentially because of the fact that in this new manner of handling the details of the solution one proceeds to find, as an intermediate step, the distribution of supersonic sources which will produce the annular body contour which will have minimum drag, rather than seeking out straight off what this shape of external contour must be.

2. Computation of the External Drag Exerted Upon the Cone-Like Duct

Examination of the more general case where $R_1 < R_2$ is to be undertaken right from the start in this instance, because the solution of the problem of finding the shape giving least drag which pertains to such a cone-like body will not present any more essential complications than will be found to arise in connection with the more restricted case where the basic shape is almost cylindrical.

In order to derive the formal expression for the external drag coefficient, the apparent mass concept will be employed as follows: Consider the mass of fluid enclosed between the planes AA_oA' and BB_oB' which are normal to the x-axis (see Fig. 2), and which contain the entrance and exit

sections of the duct whose external contour is to be determined so as to give minimum drag. One of the other two enclosing surfaces for the mass under consideration is the simple boundary marked out by a circular cylindrical surface, σ, which is to have an arbitrary radius, denoted by R', and whose axis is coincident with the x-axis. And, finally, the last enclosing surface is the exterior surface of the rigid duct itself.

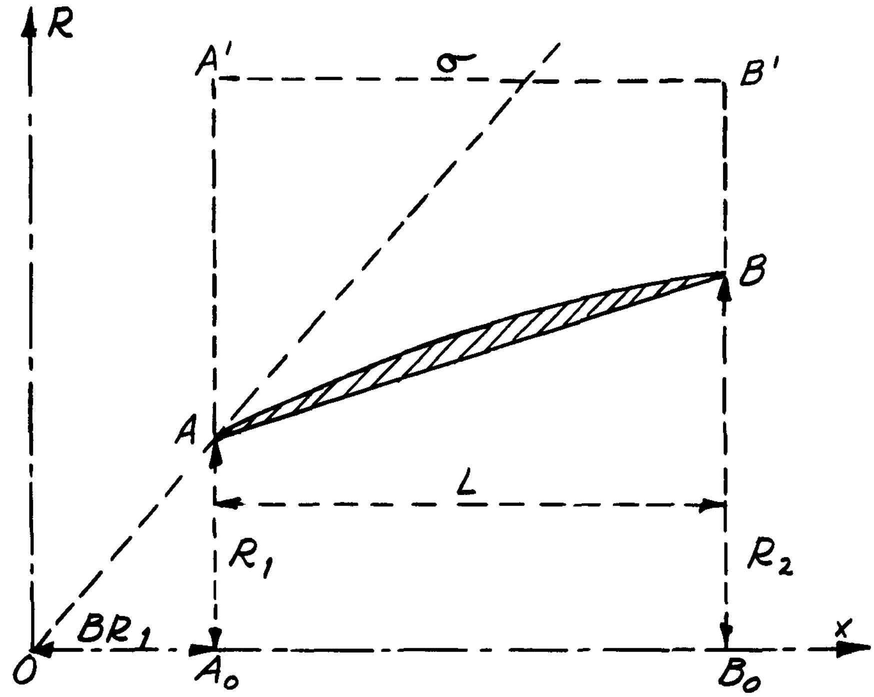


Fig. 2

Control Surfaces for Applying the Momentum Theory to Determination of the Drag Coefficient Corresponding to a Cone-Like Annular Duct

Now let Q_2 dt, Q'dt, and Q_1 dt denote the component in the direction of the x-axis of the momentum possessed by the following masses of the fluid, respectively: the mass flowing out of the section BB' per infinitesimal interval of time, dt; the mass flowing out of the cylindrical surface, σ ,

in this same time unit; and the mass entering the section AA' in this interval. In addition, also let F_D be used to represent the drag which is created in the distance between the end-points of the duct, A and B, and let F_p be taken to represent the component along the x-axis of the force which is due to the pressures acting on the sections AA' and BB'. Then, on the basis of the momentum theory, it follows that the above-defined quantities are related by the formula

$$F_D = F_p - Q_2 - Q' + Q_1 . \tag{1}$$

If one makes the further definitions that U_x and U_r are to stand for the axial and radial components of the local velocity vector, $\bar{U}$, respectively, then it may be seen that

$$Q' + Q_2 - Q_1 = 2\pi R' \int_{x_A}^{x_B} \rho U_x U_r \, dx + 2\pi \int_{R_2}^{R'} \rho R U_x^2 \, dR - \pi \rho_\infty U_\infty (R'^2 - R_1^2). \tag{2}$$

On the other hand, the equation of continuity requires under these circumstances that

$$-\rho_\infty \frac{U_\infty}{2} (R'^2 - R_1^2) + \int_{R_2}^{R'} \rho R U_x \, dR + R' \int_{x_A}^{x_B} \rho U_r \, dx = 0 . \tag{3}$$

Proceeding further with the reduction of Eq. (2), the following auxiliary definitions will also be found helpful; i.e., let

$$U_x = U_\infty + u$$

and

$$\left.\begin{aligned} \rho &\simeq \rho_\infty + \left(\frac{d\rho}{dU}\right)_{U=U_\infty} \left(U - U_\infty\right) + \frac{1}{2} \left(\frac{d^2\rho}{dU^2}\right)_{U=U_\infty} \left(U - U_\infty\right)^2 \\ &= \rho_\infty \left\{ 1 - M_\infty^2 \left(\frac{u}{U_\infty} + \frac{1}{2} \frac{U_r^2}{U_\infty^2} \right) + \frac{1}{2} \left[(2-\gamma) M_\infty^4 - M_\infty^2 \right] \frac{u^2}{U_\infty^2} . \right. \end{aligned}\right\} \tag{4}$$

Then it follows that

$$\frac{Q' + Q_2 - Q_1}{2\pi} = \rho_\infty U_\infty \int_{R_2}^{R'} \left(u - B^2 \frac{u^2}{U_\infty^2}\right) R\, dR + R' \rho_\infty \int_{x_A}^{x_B} u\, U_r \, dx \, . \quad (5)$$

Furthermore, one has that

$$-F_p = \int_{R^2}^{R'} (p - p_\infty)\, R\, dR$$

and since

$$p - p_\infty = -\rho_\infty U_\infty u - \rho_\infty \frac{u^2}{2} - \rho_\infty \frac{U_r^2}{2} + \rho_\infty M_\infty \frac{u^2}{2}$$

the expression for F_p may be put in the form

$$F_p = \rho_\infty U_\infty \int_{R_2}^{R'} R\, u\, dR - \frac{1}{2} \rho_\infty \int_{R_2}^{R'} u^2 B^2 R\, dR + \frac{1}{2} \rho_\infty \int_{R_2}^{R'} R\, U_r^2 \, dR$$

and by combining these expressions according to the dictates of Eq. (1), it is found that

$$F_D = 2\pi \int_{x_A}^{x_B} (p - p_\infty)\, R \frac{dR}{dx}\, dx$$

$$= -\rho_\infty R' \int_{x_A}^{x_B} u\, U_r \, dx + \frac{1}{2} \rho_\infty \int_{R_2}^{R'} R (B^2 u^2 + U_r^2)\, dR$$

Let it now be assumed that $R' = R_2$. That this is a legitimate simplification will be readily admitted provided the thickness (height between the annular duct's external surface and the basic internal conical surface to which the circular sections AA_0 and BB_0 belong) of the configuration under consideration is small enough. Once this agreement is settled upon, it follows immediately that F_D reduces to

$$F_D = -\rho_\infty R_2 U_\infty^2 \int_{x_A}^{x_B} \frac{u}{U_\infty} \cdot \frac{U_r}{U_\infty}\, dx$$

and thus the expression for the drag coefficient turns out to be simply

$$C_D = \frac{F_D}{\frac{1}{2}\rho_\infty \pi R_0^2 U_\infty^2} = -\frac{2}{\pi}\frac{R_2^2}{R_0^2}\int_{x_A}^{x_B} \frac{u}{U_\infty}\cdot\frac{U_r}{U_\infty}\, d\left(\frac{x}{R_2}\right).$$

Now it is well known that the u and U_r components of the flow are given by the formulae:

$$\frac{u}{U_\infty} = -\int_0^{x-BR_2} \frac{\dot{f}(\xi)\, d(\xi)}{\sqrt{(x-\xi)^2 - B^2R_2^2}}$$

and

$$\frac{U_r}{U_\infty} = \frac{1}{R_2}\int_0^{x-BR_2} \frac{(x-\xi)\dot{f}(\xi)\, d\xi}{\sqrt{(x-\xi)^2 - B^2R_2^2}}$$

where the notation is the same as that employed in the development given as Part I of this paper.

Thus, if one first lets

$$L_2 = L(1-\beta B)$$

for convenience's sake, then it follows that

$$C_D = \frac{2}{\pi}\frac{R_2^2}{R_0^2}\int_{BR_2}^{L_2+BR_2}\frac{dx}{R_2}\int_0^{x-BR_2}\frac{d\xi_1}{R_2}\int_0^{x-BR_2}\frac{\dot{f}(\xi_1)\dot{f}(\xi_2)\, d\xi_2}{\sqrt{(u-\xi_1)^2-B^2R_2^2}\sqrt{(x-\xi_2)^2-B^2R_2^2}}. \quad (6)$$

In order to put this expression into more tractable form, let the integration with respect to x be carried out first of all. The result will be

$$C_D = \frac{2}{\pi R_0^2}\int_0^{L_2}\dot{f}(\xi_1)\, d\xi_1\left[\int_{\xi_1}^{L_2}\dot{f}(\xi_2)\, d\xi_2\int_{\xi_2+BR_2}^{L_2+BR_2}\frac{(x-\xi_2)\, dx}{\sqrt{(x-\xi_1)^2-B^2R_2^2}\sqrt{(x-\xi_2)^2-B^2R_2^2}}\right.$$

$$\left. + \int_0^{\xi_1}\dot{f}(\xi_2)\, d\xi_2\int_{\xi_1+BR_2}^{L_2+BR_2}\frac{(x-\xi_2)\, dx}{\sqrt{(x-\xi_1)^2-B^2R_2^2}\sqrt{(x-\xi_2)^2-B^2R_2^2}}\right]$$

$$= \frac{2}{\pi R_o^2}\int_0^{L_2} \dot{f}(\xi_2)\, d\xi_2 \left[\int_{\xi_2}^{L_2} \dot{f}(\xi_1)\, d\xi_1 \int_{\xi_1+BR_2}^{L_2+BR_2} \frac{(x-\xi_2)\,dx}{\sqrt{(x-\xi_1)^2-B^2R_2^2}\cdot\sqrt{(x-\xi_2)^2-B^2R_2^2}}\right.$$

$$\left. + \int_0^{\xi_2} \dot{f}(\xi_1)\, d\xi_1 \int_{\xi_2+BR_2}^{L_2+BR_2} \frac{(x-\xi_2)\,dx}{\sqrt{(x-\xi_1)^2-B^2R_2^2}\cdot\sqrt{(x-\xi_2)^2-B^2R_2^2}}\right]. \qquad (6')$$

The very last integral in the above formulation may be simplified further by proceeding as follows:

Let
$$H_2(t_1,t_2) = \int_{\xi_2+BR_2}^{L_2+BR_2} \frac{(x-\xi_2)\,dx}{\sqrt{(x-\xi_1)^2-B^2R_2^2}\cdot\sqrt{(x-\xi_2)^2-B^2R_2^2}}$$

and then
$$H_2(t_1,t_2) = \int_{1+t_2}^{1+\ell_2^*} \frac{(\eta_2-t_2)\,d\eta_2}{\sqrt{(\eta_2-t_2)^2-1}\cdot\sqrt{(\eta_2-t_1)^2-1}} \qquad (7)$$

where the improvement in symmetry has been made by applying the transformation defined by

$$\left.\begin{aligned} \eta_2 &= \frac{x}{BR_2} \\ t_2 &= \frac{\xi_2}{BR_2} \\ t_1 &= \frac{\xi_1}{BR_2} \\ \text{and} \quad \ell_2^* &= \frac{L_2}{BR_2} = \ell_2\,(1-\beta B)\,. \end{aligned}\right\} \qquad (7')$$

The evaluation of $H_2(t_1,t_2)$ may be performed by recourse to elliptic integrals; i.e., one finds that

$$\left.\begin{aligned}\int_{1+t_2}^{1+\ell_2^*}\frac{(\mathfrak{z}_2-t_2)\,d\mathfrak{z}_2}{\sqrt{(\mathfrak{z}_2-t_2)^2-1}\cdot\sqrt{(\mathfrak{z}_2-t_1)^2-1}} &= \frac{2}{t_2-t_1}\left[F(\varphi_0,k)+2\Pi(\varphi_0,k,n)\right] \\ &\qquad\text{for } \frac{t_2-t_1}{2}>1 \\ &=(t_2-t_1-1)F(\varphi_1,k_1)+(t_2-t_1)\Pi(\varphi_1,k_1,n_1) \\ &\qquad\text{for } \frac{t_2-t_1}{2}<1\end{aligned}\right\}\quad(8)$$

where $F = F(\varphi_0,k)$ is the elliptic integral of the first kind with argument φ_0, which is defined as

$$\varphi_0 = \arcsin\sqrt{\frac{(\ell_2^*-t_2)(t_2-t_1)}{(\ell_2^*-t_2+2)(t_2-t_1+2)}} \qquad (9)$$

and the modulus of this elliptic integral is denoted by k , where

$$k^2 = 1-\left(\frac{2}{t_2-t_1}\right)^2. \qquad (9')$$

The other symbol employed in Eq. (8), i.e., $\Pi(\varphi_0,k,n)$, stands for the elliptic integral of the third kind, whose corresponding argument, φ_0 , and modulus, k , are given by the same formulae just set down above as Eqs. (9) and (9'), while the parameter, n , which appears in connection with this integral has the definition:

$$n = -\left(1+\frac{2}{t_2-t_1}\right). \qquad (9'')$$

Finally, the other elliptic integral of the third kind employed in Eq. (8) has an argument, modulus, and parameter which are determined according to the relationships

$$\left.\begin{aligned}\varphi_1 &= \arcsin\sqrt{\frac{2(\ell_2^*-t_2)}{(\ell_2^*-t_1)(t_2-t_1+2)}} \\ k_1^2 &= 1-\left(\frac{t_2-t_1}{2}\right)^2 \\ \text{and}\quad n_1 &= -\left(1+\frac{t_2-t_1}{2}\right).\end{aligned}\right. \qquad (10)$$

In an analogous manner, the other integral appearing in Eq. (6') may be handled by letting

$$H_1(t_1,t_2) = \int_{\xi_1+BR_2}^{L_2+BR_2} \frac{(x-\xi_2)\,dx}{\sqrt{(x-\xi_1)^2-B^2R_2^2}\cdot\sqrt{(x-\xi_2)^2-B^2R_2^2}}$$

and, by use of the transformations (7'), it then results that

$$H_1(t_1,t_2) = \int_{1+t_1}^{1+\ell_2^*} \frac{(\mathfrak{z}_2-t_2)\,d\mathfrak{z}_2}{\sqrt{(\mathfrak{z}_2-t_1)^2-1}\cdot\sqrt{(\mathfrak{z}_2-t_2)^2-1}}$$

and the evaluation of this $H_1(t_1,t_2)$ function may also be effected in terms of elliptic integrals as follows:

$$\left.\begin{aligned} H_1(t_1,t_2) &= \frac{2(t_2-t_1+1)}{t_1-t_2}F(\varphi_2,k) + \frac{4}{t_1-t_2}\Pi(\varphi_2,k,n_2) \qquad \text{for } \frac{t_1-t_2}{2} > 1 \\ &= -F(\varphi_3,k_1) + (t_1-t_2)\,\Pi(\varphi_3,k_1,n_3) \qquad \text{for } \frac{t_1-t_2}{2} < 1 \end{aligned}\right\} \quad (11)$$

where the new arguments and parameters appearing here are defined by the formulae:

$$\left.\begin{aligned} \varphi_2 &= \arcsin\sqrt{\frac{(\ell_2^*-t_1)(t_1-t_2)}{(\ell_2^*-t_1+2)(t_1-t_2+2)}} \\ \varphi_3 &= \arcsin\sqrt{\frac{2(\ell_2^*-t_1)}{(\ell_2^*-t_2)(t_1-t_2+2)}} \\ n_2 &= -\left(1+\frac{2}{t_1-t_2}\right) \\ \text{and}\qquad n_3 &= -\left(1+\frac{t_1-t_2}{2}\right) \end{aligned}\right\} \quad (12)$$

while the modulii k and k_1 are determined by the same formulae as used in the previous case.

If one now scrutinizes carefully the expressions given for the arguments and parameters in Eqs. (9), (10), and (12), it will be seen that the following interrelationships exist between the later set and the ones given previously:

$$\left.\begin{aligned} \varphi_2(t_1, t_2) &= \varphi_0(t_2, t_1) \\ \varphi_3(t_1, t_2) &= \varphi_1(t_2, t_1) \\ \text{and} \quad n_2(t_1, t_2) &= n(t_2, t_1) \\ n_3(t_1, t_2) &= n_1(t_2, t_1). \end{aligned}\right\} \quad (13)$$

Consequently, the expression for the drag coefficient given as Eq. (6') reduces now to

$$C_D = \frac{2}{\pi} B^2 \frac{R_2^2}{R_0^2} \int_0^{\ell_2^*} \dot{f}(t_1)\, dt_1 \int_0^{\ell_2^*} \dot{f}(t_2)\, H(t_1, t_2)\, dt_2$$

$$= \frac{2}{\pi} B^2 \frac{R_2^2}{R_0^2} \int_0^{\ell_2^*} \dot{f}(t_2)\, dt_2 \int_0^{\ell_2^*} \dot{f}(t_1)\, H(t_1, t_2)\, dt_1 \quad (14)$$

where the understanding is made that the interval of integration from 0 to ℓ_2^*, both in the case where the integrand is $\dot{f}(t_2)H(t_1,t_2)$ as well as in the case where it is $\dot{f}(t_1)H(t_1,t_2)$, will have to be divided into two sub-intervals, which will run from 0 to t_1 and then from t_1 to ℓ_2^* in the case of the first expression given for C_D, while in the second expression, the splitting up of the 0 to ℓ_2^* interval will be done at the point t_2, so that the two sub-intervals run in this case from 0 to t_2 and then from t_2 to ℓ_2^*. Of course, in each case the H function to be used will depend on whether t_1 is less or greater than t_2; i.e., H will be set equal to H_1 if $t_2 < t_1$, while it will be replaced by the H_2 expression in the part of the interval of integration where $t_2 > t_1$.

3. Determination of How the Supersonic Source Distribution Must Vary Along the x-Axis in Order to Produce a Minimum Drag, $(C_D)\min$

The statement of the boundary condition for the external flow over the duct in question results in the well-known relationship

$$\frac{dR_e}{dx} \simeq \frac{1}{R_1+\beta(x-BR_1)}\int_0^{x-BR} \frac{(x-\xi)\,\dot{f}(\xi)\,d\xi}{\sqrt{(x-\xi)^2-B^2[R_1+\beta(u-BR_1)]^2}}$$

where, for brevity's sake, the upper limit on the integral has merely been denoted by x - BR. This condition may be improved slightly in form by making the substitution $\omega = R_1/R_2$, to give

$$\frac{dR_e}{dx} = \frac{B}{\omega+\beta B(\eta_2-\omega)}\int_0^{(\eta_2-\omega)(1-\beta B)} \frac{(\eta_2-t_2)\,\dot{f}(t_2)\,dt_2}{\sqrt{(\eta_2-t_2)^2-[\omega+\beta B(\eta-\omega)]^2}} \qquad (15)$$

and consequently a more directly useful form of this boundary condition results as

$$\frac{R_e-R_1}{BR_2} = B\int_\omega^{\eta_2} \frac{dt_2}{\omega+\beta B(t_1-\omega)}\int_0^{(t_1-\omega)(1-\beta B)} \frac{(t_1-t_2)\,\dot{f}(t_2)\,dt_2}{\sqrt{(t_1-t_2)^2-[\omega+\beta B(t_1-\omega)]^2}}\,. \qquad (16)$$

If the order of integration in the above expression is inverted, it then may be written as

$$\frac{R_e-R_1}{BR_2} = B\int_0^{(\eta_2-\omega)(1-\beta B)} \dot{f}(t_2)\,dt_2 \int_{\frac{t_2}{1-\beta B}+\omega}^{\eta_2} \frac{(t_1-t_2)\,dt_1}{[\omega+\beta B(t_1-\omega)]\sqrt{(t_1-t_2)^2-[\omega+\beta B(t_1-\omega)]^2}}\,. \qquad (16')$$

The next step in rendering this expression into more suitable form for later

manipulation is advanced by making the following substitutions:

$$\left.\begin{aligned} t_1^* &= \omega + \beta B\,(t_1 - \omega) \\ \text{and} \quad t_1 - t_2 &= \frac{t_1^*}{\beta B} - \omega \cdot \frac{1-\beta B}{\beta B} - t_2 = \frac{t_1^*}{\beta B} - Y \end{aligned}\right\} \tag{17}$$

where
$$Y = t_2 - \omega \cdot \frac{1-\beta B}{\beta B} \tag{17'}$$

and thus the integral with respect to t_1 indicated in Eq. (16') may be recast into the form

$$\begin{aligned} I &= \int_{\frac{t_2}{1-\beta B}+\omega}^{\vartheta_2} \frac{(t_1 - t_2)\,dt_1}{[\omega+\beta B(t_1-\omega)]\sqrt{(t_1-t_2)^2 - [\omega+\beta B(t_1-\omega)]^2}} \\ &= \frac{1}{\beta B} \int_{\frac{\beta B Y}{1-\beta B}}^{\beta B \psi} \frac{dt_1^*}{\sqrt{t_1^{*2}(1-\beta^2B^2) - 2\,\beta B Y\, t_1^* + \beta^2 B^2 Y^2}} \\ &= -Y \int_{\frac{\beta B Y}{1-\beta B}}^{\beta B \psi} \frac{dt_1^*}{t_1^* \sqrt{t_1^{*2}(1-\beta^2B^2) - 2\,\beta B Y t_1^* + \beta^2 B^2 Y^2}} \\ &= I_1 + I_2 \end{aligned} \tag{18}$$

where $\psi = \vartheta_2 - \omega \cdot \dfrac{1-\beta B}{\beta B}$.

It will be clear that the evaluation of the integrals I_1 and I_2 results in the following explicit expressions for them:

$$\left.\begin{aligned} I_1 &= \frac{1}{\beta B} \cdot \frac{1}{\sqrt{1-\beta^2B^2}} \log \frac{(1-\beta^2B^2)\,\psi - Y + \sqrt{1-\beta^2B^2}\sqrt{(\psi - Y)^2 - \beta^2B^2\psi^2}}{\beta B Y} \\ \text{and} \quad I_2 &= \frac{1}{\beta B} \log \frac{\psi - Y - \sqrt{(\psi - Y)^2 - \beta^2B^2\psi^2}}{\beta B \psi} \end{aligned}\right\} \tag{18'}$$

and consequently the sought expression, to be used in expressing the peri-

metral condition to be applied below, takes the form

$$\frac{R-R_1}{BR_2} = B\int_0^{(z_2-\omega)(1-\beta B)} I(z_2,t_2)\,\dot{f}(t_2)\,dt_2 \tag{19}$$

where the indicated $I(z_2,t_2)$ function is the one expressed as Eq. (18').

Now the desired expression for the integral of this thickness parameter $\frac{R-R_1}{BR_2}$ is given in extended form as

$$\int_{BR_1}^{BR_1+L} \frac{R-R_1}{BR_2}\,dx = B^2R_2\int_{\omega}^{\omega+\ell_2} dz_2 \int_0^{(z_2-\omega)(1-\beta B)} I(z_2,t_2)\,\dot{f}(t_2)\,dt_2$$

$$= B^2R_2\int_0^{\ell_2^*} \mathcal{Z}(t_2)\,\dot{f}(t_2)\,dt_2$$

where

$$\mathcal{Z}(t_2) = \int_{\frac{t_2}{1-\beta B}+\omega}^{\omega+\frac{\ell_2^*}{1-\beta B}} I(z_2,t_2)\,dz_2 . \tag{20}$$

Finally, therefore, the statement of the (a)-type of perimetral condition discussed in Article 1 of Part I may be put into the simple form

$$\int_0^{\ell_2^*} \mathcal{Z}(t_2)\,\dot{f}(t_2)\,dt_2 = C . \tag{21}$$

In an entirely analogous manner the other "volume" perimetral condition may be given a simple formulation by first noting that

$$\int_{BR_1}^{BR_1+L} (R_e^2-R_i^2)\,dx \cong \frac{1}{2}\int_{BR_1} \left[R_1+\beta(x-BR_1)\right]\left[R_e-R_1-\Delta R\right] dx .$$

where once again the average duct radius is expressed as

$$\frac{R_e+R_i}{2} = R_1+\beta(x-\beta R_1)$$

and the internal radius is given in terms of the mouth radius by the relation

$$R_i = R_1 + \Delta R \, .$$

Consequently, the (b)-type of perimetral condition discussed in Article 1 of Part I may be expressed in the form

$$\int_{\omega}^{\omega+\frac{L}{BR_2}} \left[\omega(1-\beta B)+\beta B \zeta_2\right] d\zeta_2 \int_0^{(\zeta_2-\omega)(1-\beta B)} I(\zeta_2, t_2)\,\dot{f}(t_2)\,dt_2$$

$$= \int_0^{\ell_2^*} G^*(t_2)\,\dot{f}(t_2)\,dt_2 = D \tag{22}$$

provided it is understood that $G^*(t_2)$ stands for

$$G^*(t_2) = \int_{\frac{t_2}{1-\beta B}+\omega}^{\frac{\ell_2^*}{1-\beta B}+\omega} \left[\omega(1-\beta B)+\beta B \zeta_2\right] I(\zeta_2, t_2)\,d\zeta_2 \, . \tag{22'}$$

Of course, the final "closure" condition to be satisfied in all events is that

$$\frac{R_2 - R_1}{BR_2} = \frac{1}{B}(1-\omega) = B\int_0^{\ell_2^*} I\left(\frac{\ell_2^*}{1-\beta B}+\omega, t_2\right)\dot{f}(t_2)\,dt_2 = \text{constant.} \tag{24}$$

The first variation in the drag coefficient C_D which results from a variation in $\dot{f}(t_2)$, denoted by $\Delta \dot{f}(t_2)$, will be seen, by reference to Eq. (14), to be expressed by the following equation

$$\Delta C_D = \frac{2B^2}{\pi}\frac{R_2^2}{R_0^2}\left[\int_0^{\ell_2^*}\Delta\dot{f}(t_2)\,dt_2\int_0^{\ell_2^*}\dot{f}(t_2)H(t_1,t_2)\,dt_2 + \int_0^{\ell_2^*}\Delta\dot{f}(t_2)\,dt_2\int_0^{\ell_2^*}\dot{f}(t_1)H(t_1,t_2)\,dt_1\right]$$

$$= \frac{2B^2}{\pi}\frac{R_2^2}{R_0^2}\left\{\int_0^{\ell_2^*}\Delta\dot{f}(t_1)\,dt_1\int_0^{\ell_2^*}\dot{f}(t_2)\left[H(t_1,t_2)+H(t_2,t_1)\right]dt_2\right\} \tag{25}$$

By remembering how the expressions for H_1 and H_2 were given further elaboration in Eqs. (7), (8), and so on, it is of aid for the present purposes to realize that

$$
\left.
\begin{aligned}
&\underline{\text{In the case where } t_1 \leq t_2 \leq \ell_2^*:}\\
&H(t_1,t_2)+H(t_2,t_1) = H_2(t_1,t_2)+H_1(t_2,t_1)\\
&\qquad = \frac{4+2(t_1-t_2)}{t_2-t_1}F(\varphi_0,k)+\frac{8}{t_2-t_1}\Pi(\varphi_0,k,n) \qquad \text{for } \frac{t_2-t_1}{2} > 1\\
&\qquad = (t_2-t_1-2)F(\varphi_1,k_1)+2(t_2-t_1)\Pi(\varphi_1,k_1,n_1) \qquad \text{for } \frac{t_2-t_1}{2} > 1\\
&\underline{\text{In the case where } 0 \leq t_2 \leq t_1:}\\
&H(t_1,t_2)+H(t_2,t_1) = H_1(t_1,t_2)+H_2(t_2,t_1)\\
&\qquad = \frac{2(t_2-t_1)+4}{t_1-t_2}F(\varphi_2,k)+\frac{8}{t_1-t_2}\Pi(\varphi_2,k,n_2) \qquad \text{for } \frac{t_1-t_2}{2} > 1\\
&\qquad = (t_1-t_2-2)F(\varphi_3,k_1)+2(t_1-t_2)\Pi(\varphi_3,k_1,n_3) \qquad \text{for } \frac{t_1-t_2}{2} < 1.
\end{aligned}
\right\} \quad (26)
$$

Recognizing that the $H(t_1,t_2)+H(t_2,t_1)$ function just discussed will take on the alternative expressions given above in the separate parts of the interval from 0 to ℓ_2^*, let the convention now be made that this function is to be denoted more simply by the single symbol $M(t_1,t_2)$.

Consequently, the first variation in the drag coefficient may be written as

$$\Delta C_D = \frac{2B^2}{\pi} \frac{R_2^2}{R_0^2} \int_0^{\ell_2^*} \Delta \dot{f}(t_1) \int_0^{\ell_2^*} \dot{f}(t_2) M(t_1, t_2)\, dt_2 = 0 \tag{27}$$

provided that the $\dot{f}(t_2)$ function really defines the way the distribution of supersonic sources must vary along the x-axis in order for the corresponding drag to be a minimum, where this minimum is denoted by the symbol $(C_D)_{min}$.

Since it is required that the conditions expressed as Eq. (21) [or else by Eq. (22)] in conjunction with Eq. (24) are to be satisfied at all times, it follows that it will have to be true that

$$\int_0^{\ell_2^*} \Delta \dot{f}(t_1)\, I\left(\frac{\ell_2^*}{1-\beta B} + \omega, t_1\right) dt_1 = 0$$

and
$$\int_0^{\ell_2^*} \Delta \dot{f}(t_1)\, G(t_1)\, dt_1 = 0$$

must hold for any such variation $\Delta \dot{f}(t_1)$.

It will have to be true, consequently, that the function $\dot{f}(t_2)$ must satisfy the relationship

$$\int_0^{\ell_2^*} \dot{f}(t_2) M(t_1, t_2)\, dt_2 + \lambda_1 I\left(\frac{\ell_2^*}{1-\beta B} + \omega, t_1\right) + \lambda_2 G(t_1) = a \tag{28}$$

On the other hand, if it is condition (22), instead of condition (21), which is required to be satisfied, an equation exactly similar to this latter Eq. (28) will have to hold, where the new condition on $\dot{f}(t_2)$ will be obtained from the previous one by the mere substitution of the function $G^*(t_1)$ in place of the function $G(t_1)$ that appears here in Eq. (28).

4. Solution of the Integral Equation, Eq. (28).

The determining equation for the required source distribution which has now been derived as Eq. (28) is an integral equation of the first kind, and, as is well known, such an equation does not have a solution in general, if the given function is arbitrary. The investigation carried out in Part I of this paper, however, showed that there does actually exist a function which describes the external contour of the duct in such wise that the drag produced by this duct shape is a minimum. In addition to acknowledgment of this fact, it may also be observed that to each such external contour, described by any such meridional line, there corresponds one and but one single distribution of supersonic source strengths along the x-axis, denoted by $f(t)$, as may be seen by referring back to what has already been deduced in Article 6 of Part I. Upon realization of the import of these facts, therefore, it may be concluded, even without the necessity of carrying out the proof starting from first principles, which would be rather laborious, that there does exist a solution to Eq. (28) and that it will be unique.

Before going on to the determination of the solution, which one is assured exists uniquely in light of what has just been said, it is worth pausing to observe that the kernel of the integral equation, $M(t_1,t_2)$, exhibits a logarithmic singularity at the point where $t_1 = t_2$. In more precise analytic terms it may be stated that

$$M(t_1, t_1+\delta) = \log|\delta| + O(1) + O(\delta) \tag{29}$$

where the symbol $|\delta|$ is used to denote the absolute difference between the t-values, i.e., $|t_2 - t_1| = |\delta|$.

Let the integral equation under consideration be rewritten, therefore, in the form

$$\int_0^{\ell_2^*} \dot{f}(t_2)\,\frac{M(t_1,t_2)}{\log|t_2-t_1|}\,\log|t_2-t_1|\,dt_2+\lambda_1\,I\left(\frac{\ell_2^*}{1-\beta B}+\omega,t_1\right)+\lambda_2\,\mho(t_1)=0. \tag{28'}$$

As a result of this manipulation, therefore, the function $\frac{M(t_1,t_2)}{\log|t_2-t_1|}$ then remains finite throughout the whole interval running from 0 to ℓ_2^*, and it takes on the value unity at the place where $t_2 = t_1$.

In order to obtain an approximate solution to Eq. (28') one may proceed by first dividing up the interval from 0 to ℓ_2^* into m sub-intervals of equal length to be represented by the symbol a. In addition, the function $\frac{M(t_1,t_2)}{\log|t_2-t_1|}$ is to be replaced by a step-function (holding for any arbitrary general value of t_1 whatsoever) which may be set up as follows:

Let $t_1^{(i)}$ represent the value of t_1 which corresponds to the midpoint of the i-th sub-interval, and let $t_2^{(j)}$ represent the value of t_2 which corresponds to the midpoint of the j-th sub-interval. Then the definition is made that

$$N\left(t_1^{(i)}, t_2^{(j)}\right)=\frac{M\left(t_1^{(i)}, t_2^{(j)}\right)}{\log\left|t_1^{(i)}-t_2^{(j)}\right|}. \tag{30}$$

Now in each one of the m small sub-intervals the value of $\dot{f}(t_2)$ is taken to be constant and to have the value which this $\dot{f}(t_2)$ function attains at the midpoint of the small sub-interval under examination. If one then represents these constants in the form

$$A_j=\dot{f}\left(t_2^{(j)}\right) \tag{31}$$

it follows that the integral appearing in Eq. (28') may be represented by the following sum:

$$\int_0^{\ell_2^*} \dot{f}(t_2)\,\frac{M(t_1^{(i)},t_2)}{\log\left|t_1^{(i)}-t_2\right|}\,\log\left|t_1^{(i)}-t_2\right|\,dt_2$$

$$=\sum_{j=1}^{m}{}' A_j\,N(t_1^{(i)},t_2^{(j)})\,R(t_1^{(i)},t_2^{(j)})+\tfrac{1}{2}\,a\left[\log\left(\tfrac{a}{2}\right)-1\right]A_i$$

where $R(t_1^{(i)},t_2^{(j)})=\displaystyle\int_{t_2^{(j)}-\frac{a}{2}}^{t_2^{(j)}+\frac{a}{2}}\log\left|t_2-t_2^{(i)}\right|\,dt_2$

$$=\left(\left|t_2^{(j)}-t_1^{(i)}\right|+\tfrac{a}{2}\right)\log\left(\left|t_2^{(j)}-t_1^{(i)}\right|+\tfrac{a}{2}\right)$$

$$-\left(\left|t_2^{(j)}-t_1^{(i)}\right|-\tfrac{a}{2}\right)\log\left(\left|t_2^{(j)}-t_1^{(i)}\right|-\tfrac{a}{2}\right)-a \tag{32}$$

for $j \neq i$.

If one now imposes the condition that Eq. (28') must be satisfied at all m midpoints of the individual m sub-intervals just defined above, the following system of m equations in the m unknowns, A , will thus result:

$$\sum_{j=1}^{m}{}' N\left(t_1^{(i)},t_2^{(j)}\right)R\left(t_1^{(i)},t_2^{(j)}\right)A_j+\tfrac{1}{2}\,a\left[\log\tfrac{a}{2}-1\right]A_i$$
$$+\lambda_1 I\left(\frac{\ell_2^*}{1-\beta B}+\omega,t_1^{(i)}\right)+\lambda_2\,\mathcal{Q}(t_1^{(i)})=0 \tag{33}$$

for $i = 1,2,\cdots,m.$

In conjunction with the condition (33) it is also necessary to have the following two conditions satisfied:

$$\left.\begin{aligned} &\sum_{j=1}^{m} I\left(\frac{\ell_2^*}{1-\beta B} + \omega, t_2^{(j)}\right) a A_j && = \text{constant} \\ \text{and} \quad &\sum_{j=1}^{m} \mathcal{Y}\left(t_2^{(j)}\right) a A_j && = \text{constant.} \end{aligned}\right\} \quad (34)$$

Thus Eqs. (33) and (34) constitute a system of $m+2$ equations in the $m+2$ unknowns A, λ_1, and λ_2, and thus they afford the means of arriving at the solution to the problem in hand.

It should be pointed out at this juncture that on the basis of the definition of the function $M(t_1, t_2)$ that is given as Eq. (26) above, and if one also recalls how the values for φ_0 and φ_2 were originally set up by means of Eqs. (9) and (12), it will be readily appreciated that when the i-subscript and j-subscript are widely different, then the value of $M(t_1^{(i)}, t_2^{(j)})$ will lie close to zero for large values of $\left| t_1^{(i)} - t_2^{(j)} \right|$. Consequently, it then becomes abundantly clear that some of the coefficients contained in the set A_j appearing in Eq. (33) overshadow those for which the i-subscripts are close to the j-subscript. This interesting property, which is an essential characteristic of the A_j-values, will stand in good stead when it comes to solving the system of equations written out explicitly above as Eq. (33), because the method of iteration which would be employed in carrying out this sort of computation may thus be short-circuited or telescoped in great degree.

One should not fail to observe the very significant fact, finally, that the parameter n, which is one of the factors which identifies and specifies concretely the elliptic integral of the third kind that enters into the definition of the M function under consideration, always takes on values that

are less than -1 in the present instance. The importance of this statement lies in the realization that one may thus proceed to evaluate this elliptic integral in an especially easy way, i.e., by utilization of the simple equivalent formula established by Hüel[8]. In order to evaluate this elliptic integral under the present circumstances, therefore, it is merely necessary to obtain, as a preliminary step, the values of α and m from the following formulations:

$$n = -\frac{1}{sn^2(\alpha, k)}$$

and

$$m = \frac{cn(\alpha, k)\cdot dn(\alpha, k)}{sn(\alpha, k)} \tag{35}$$

where sn, cn, and dn are the Jacobi elliptic functions known more precisely as the sine-amplitude, cosine-amplitude, and delta-amplitude functions. Furthermore, it is then useful to define quantities q and Λ according to the scheme

$$q = \frac{1}{2}\frac{\alpha}{F}$$

and

$$\Lambda = \frac{F(\varphi, k)}{2F}$$

so that finally the evaluation of $\Pi(\varphi, n, k)$ is obtained from the relation:

$$m\,\Pi(\varphi, n, k) = \frac{1}{2}\log\frac{\vartheta_1(\Lambda + q)}{\vartheta_1(\Lambda - q)} - \frac{\vartheta_4'(q)}{\vartheta_4(q)}\Lambda \tag{35'}$$

where ϑ denotes the Jacobi theta function, and, of course, as previously, F represents the complete elliptic integral of the first kind with modulus k, while $F(\varphi, k)$ is the elliptic integral of the first kind with argument φ and modulus k.

Fortunately, therefore, the evaluation for the elliptic integral of the third kind appearing in Eq. (26) may thus be carried out by means of Eq. (35') merely through reference to standard tables.

5. Determination of the Meridional Profile for the Annular Duct Which Will Have the Minimum Amount of Drag

Once the distribution of supersonic sources located along the x-axis has been obtained as explained above, then the determination of the shape which the corresponding external duct contour will assume, to produce least drag under the stipulated conditions, will require but a slight amount of further treatment of the information already at hand. As a matter of fact, all that is necessary is to make use of the expression given previously as Eq. (19) for the thickness parameter, $\frac{R_e - R_1}{BR_2}$.

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